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Groups Generated by Two Operators Whose Relative Transforms are Equal to Each Other.

By G. A. Miller.

If s and t represent two operators of a group the important operator

$$s^{-1}ts$$

is known as the *transform* of t with respect to s. Similarly $t^{-1}st$ is called the transform of s with respect to t. These two transforms are usually distinct. When s and t are commutative they are equal to each other whenever s and t are identical and only then. When s and t are non-commutative the equality of these transforms still implies, among other things, that s and t have the same order, since the order of an operator is equal to that of all its transforms. The main object of the present paper is to determine fundamental properties of the groups generated by s and t when they satisfy the equation

$$s^{-1}ts=t^{-1}st, \qquad \text{(A)}$$

but are not otherwise restricted.

Our first object is to determine the order of $t^{-1}s$ when the common order of s and t is an arbitrary positive integer α. For the determination of this order it is convenient to employ the equation

$$(t^{-1}s)^2=st^{-1}, \qquad \text{(B)}$$

which can readily be established by writing (A) in the form

$$t^{-1}s=s^{-1}tst^{-1},$$

and substituting in the second member of

$$(t^{-1}s)^2=t^{-1}st^{-1}s.$$

From (B) it results directly that $t^{-1}s$ is transformed into its square by s^{-1}. Hence s transforms into itself the cyclic group generated by $t^{-1}s$, and the group G generated by this cyclic group and s is identical with the group generated by s and t. In particular, it results that G is of finite order provided $t^{-1}s$ is of finite order. We shall now prove that the latter order is a divisor of

$$2^\alpha-1=2^{\alpha-1}+2^{\alpha-2}+\ldots+2+1.$$

This fact can easily be established by means of (B). When $\alpha=1$ no proof is required. When $\alpha>1$, sets of four factors in the first member of the following equation, beginning at the left,

$$t^{-1}st^{-1}s\ldots t^{-1}s=(t^{-1}s)^{2^\alpha-1}$$

can be replaced by sets of two factors by means of (B). We thus obtain the simpler equation

$$st^{-1}st^{-1}\ldots t^{-2}s=(st^{-1})^{2^{\alpha-1}-1}\cdot t^{-1}s=s(t^{-1}s)^{2^{\alpha-1}-2}\cdot t^{-2}s.$$

When $\alpha=2$ it is evident that the members of the last equation reduce to identity. When $\alpha>2$ we may write out the last member of this equation and in it affect a similar reduction by means of (B), thus obtaining the equation

$$s^2t^{-1}st^{-1}s\ldots t^{-1}s=s^2(t^{-1}s)^{2^{\alpha-2}-2}\cdot t^{-3}s.$$

The members of this equation are evidently equal to the identity when $\alpha=3$. When $\alpha>3$ the process may clearly be repeated until the identity is reached. Hence the theorem:

If s and t are two operators of order α which satisfy the condition $s^{-1}ts=t^{-1}st$, then the order of $t^{-1}s$ is a divisor of $2^\alpha-1$.

The fact that the order of $t^{-1}s$ must be exactly $2^\alpha-1$ whenever no additional restriction is placed on s and t, may be established by actually constructing a group involving operators which satisfy these conditions. We proceed to do this. Let t_1 represent an operator of order $2^\alpha-1$ and let s_1 represent an operator of order α in the group of isomorphisms of the cyclic group generated by t_1. Since 2 belongs to exponent α mod $2^\alpha-1$ it may be assumed that $s_1t_1s_1^{-1}=t_1^2$, and hence

$$t_1s_1^{-1}t_1^{-1}=s_1^{-1}t_1.$$

The operators s_1 and $s_1t_1^{-1}$ are therefore of order α. Their relative transforms are equal to each other since

$$s_1^{-1}\cdot s_1t_1^{-1}s_1=t_1^{-1}s_1 \text{ and } t_1s_1^{-1}\cdot s_1\cdot s_1t_1^{-1}=t_1s_1t_1^{-1}.$$

In fact, the second members of these two equations are the inverses of operators which were proved equal to each other and hence they must also be equal to each other.

From the preceding paragraph it results that there are two operators of order α, α being any positive integer, whose relative transforms are equal to each other and which satisfy the condition that the product of one and the inverse of the other is of order $2^\alpha-1$. Hence the following theorem has been established:

If the relative transforms of two operators of a group are equal to each other these operators have the same order α and when they are not otherwise restricted they generate a solvable group of order $\alpha\ (2^{\alpha}-1)$.

For a particular value of α there is one and only one such group. This group contains a cyclic commutator subgroup of order $2^{\alpha}-1$ and is generated by this subgroup and an operator of order α which transforms each of its operators into its square. In view of the great importance of the concept of transform in the theory of groups and the remarkable simplicity of these groups it may be desirable to denote this system by a special name. We shall call it the *equi-transform system* of groups.

For the sake of illustrations it may be desirable to note that when $\alpha=1$ this group is the identity, when $\alpha=2$ it is the dihedral group of order 6, when $\alpha=3$ it is the semi-metacyclic group of order 21, etc. It is evident that each of the groups in the equi-transform system contains a set of $2^{\alpha}-1$ conjugate cyclic subgroups of order α which are therefore separately transformed into themselves by only their own operators. Moreover, when α is even and represented by $2n$ all the $2^{n}+1$ operators of order 2 contained in such a group constitute a single set of conjugates. The fact that each of these operators of order 2 appears in at least one set of independent generators of G results directly from the following evident theorem:

If each one of a complete set of conjugate operators of a group can be transformed into each of the other operators of the set by some operator of the set then each of these operators appears in at least one set of independent generators of the group.

From the preceding paragraph it results directly that the ϕ-subgroup of G must always be of odd order. We proceed to prove that the order of this subgroup is $2^{\alpha}-1$ divided by the product of all its distinct prime factors. To prove this theorem it is only necessary to prove that every operator of G which is not found in its invariant cyclic subgroup of order $2^{\alpha}-1$ does appear in at least one of the sets of independent generators of G. If this theorem were not true there would be such an operator S of prime order p which would not belong to a set of conjugates having the properties noted in the theorem closing the preceding paragraph.

Since S would be a power of an operator s of order α contained in G it may be assumed that $\alpha=kp$ and that $s^{-1}ts=t^{2}$, t being a generator of the invariant cyclic subgroup of order $2^{\alpha}-1$ contained in G. Hence

$$S^{-1}tS=t^{2^k}.$$

It may be noted that when 2^a-1 is prime to p the theorem is evident. Hence we shall assume in what follows that 2^a-1 is divisible by p. Moreover, it may be assumed that S is commutative with the Sylow subgroups whose orders are prime to p contained in the invariant cyclic subgroup of order 2^a-1. Hence it results that

$$2^k-1\equiv 0 \mod (2^a-1)/p.$$

It is easy to prove that this congruence is impossible. In fact,

$$p(2^k-1)<2^a-1$$

since $p\cdot 2^k<2^a=2^{kp}$. Hence the operator S belongs to a set of conjugates such that each of the operators of the set is transformed into every operator of the set by some operator of the set, and the following theorem has been established:

The Φ-subgroup of every group in the equi-transform system is cyclic and of order 2^a-1 *divided by the product of the distinct prime factors of this number.* Incidentally it has also been proved that *every prime factor of a is a divisor of the remainder obtained by subtracting unity from some prime factor of* 2^a-1.

UNIVERSITY OF ILLINOIS.

A Classification of General (2, 3) Point Correspondences Between Two Planes.

By Temple Rice Hollcroft.

§1. Introduction and General Discussion.

1. *Introduction.*—The purpose of this paper is to discuss and classify algebraic (2, 3) point correspondences between two planes. Such a correspondence is established between two planes (x) and (x') when an algebraic relation in x_1, x_2, x_3 and x'_1, x'_2, x'_3 exists such that to any point of (x) shall correspond two points of (x'), and to any point of (x') three points of (x).

While the theory of (1, 1) correspondences has been thoroughly developed and that of $(1, n)$ correspondences treated in some detail, there are but few writings that consider the case in which both planes are multiple and thus have a direct bearing on the subject of the present paper. The first paper on (2, 2) point correspondences was published in 1889 by P. Visalli.* In this paper he discusses a very special case in which the lines of either plane correspond to conics in the other. Burali-Forti† later obtained certain (2, 2) correspondences by combining two (1, 2) correspondences and showed that the case treated by Visalli is included in these. The first investigation of general (2, 2) point correspondences was made by G. Marletta.‡ He considers two special types and deduces the properties of their associated transformations geometrically in four dimensional space. Finally an exhaustive classification of (2, 2) point correspondences has been recently made by F. R. Sharpe and Virgil Snyder.§

The only paper to date on correspondences of multiplicities greater than two is that of Richard Baldus,|| who has investigated certain properties of (m_1, m_2) point correspondences by geometrical methods.

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* P. Visalli, "La trasformazione quadratica (2, 2)," *Rend. del Circ. Mat. di Palermo*, Vol. III (1889), pp. 165–170.

† C. Burali-Forti, "Sulle trasformazione (2, 2) che si possono ottenere mediante due trasformazioni doppie," *Rend. del Circ. Mat. di Palermo*, Vol. V (1891), pp. 91–99.

‡ G. Marletta, "La trasformazione quadratica (2, 2) fra piani," *Rend. del Circ. Mat. di Palermo*, Vol. XVII (1903), pp. 173–184. "La trasformazioni cubiche (2, 2) fra piani," same volume, pp. 371–385.

§ F. R. Sharpe and V. Snyder, "Types of (2, 2) Point Correspondences Between two Planes." *T. A. M. S.*, Vol. XVIII (1917), pp. 402–414.

|| R. Baldus, "Zur Theorie der gegenzeitig mehrdeutigen algebraischen Ebenentransformationen," *Mat. Annal.*, Vol. LXXII (1912), pp. 1–56.

2. A general (2, 3) point correspondence is said to exist between two planes (x) and (x') when the points of the two planes are related as follows: Choosing (x) and (x') respectively for the double plane and the triple plane, to a point P' of (x') correspond three points P_1, P_2, P_3 of (x), ordinarily distinct. To P_1, P_2, P_3 correspond in (x') the original point P' and the residual points P'_1, P'_2, P'_3, respectively, which are usually distinct from each other and from P'. One more step will fully illustrate the general procedure. To P'_1, P'_2, P'_3, respectively, correspond the three sets P_1, P_4, P_5; P_2, P_6, P_7; P_3, P_8, P_9, the three points in each set being usually non-coincident.

If in (x') the three residual points P'_1, P'_2, P'_3 coincide in one point P'_0, then the three points P_1, P_2, P_3 of (x) correspond to either P' or P'_0 of (x') and to each of the points P_1, P_2, P_3 corresponds the pair P', P'_0. Such an involutorial (2, 3) point correspondence may be obtained by combining a (1, 2) and a (1, 3) point correspondence and is known as a (2, 3) compound involution. Since (2, 3) compound involutions, although distinct types, are not general (2, 3) point correspondences, they will not be discussed in this paper.

In the general (2, 3) point correspondence between two planes, a type is distinguished and defined by two equations, each algebraic in the coordinates of both planes. As shown in the next article, such a pair of equations establishes an algebraic relationship between the planes (x) and (x') such that to a point of (x) correspond two points of (x'), and to a point of (x') correspond three points of (x). By analogy to Riemann surfaces, (x) is called the double plane and (x') the triple plane. By considering all the possible forms of these defining equations that do not establish (2, 3) compound involutions, twelve independent types of general (2, 3) point correspondences have been obtained. By independent types we mean those that can not be reduced, one to the other, by any series of birational transformations. It will be shown finally that all general (2, 3) point correspondences are birationally equivalent to some one of these twelve types.

3. *General Properties.*—The defining equations of a (2, 3) point correspondence between two planes (x) and (x') may be written as two algebraic equations of the form,

$$\Sigma u_k(x)\, u'_k(x') = 0, \tag{1}$$

$$\Sigma v_k(x)\, v'_k(x') = 0. \tag{2}$$

When x'_1, x'_2, x'_3 are parameters, these equations represent two curves of (x) intersecting in three variable points corresponding to the point (x'_1, x'_2, x'_3),

and when x_1, x_2, x_3 are parameters, two curves of (x') intersecting in two variable points corresponding to the point (x_1, x_2, x_3). Connected with each (2, 3) point correspondence, the defining equations establish a transformation between the two planes (x) and (x') such that to a line $C_1(x)$ corresponds a curve $C'_n(x')$ of order n and genus p' which may have fundamental points A'_j of multiplicities r'_j. Likewise to a line $C'_1(x')$ corresponds a curve $C_n(x)$ of order n and genus p which may have fundamental points A_i of multiplicities r_i.

The fundamental or basis points of a given plane are fixed points on the curves of that plane. The order of a basis point is its multiplicity on the images of general lines. To each basis point of order r in either plane corresponds a basis curve of order r in the other plane. If a curve C passes through a basis point A its image is composite, consisting of the image of A counted as many times as the multiplicity of A on C and a residual curve called the proper image of C.

To a point P' of (x') correspond three points P_1, P_2, P_3 of (x) which describe the curve C_n as P' describes the line C'_1. Similarly, to a point P of (x) correspond two points P'_1, P'_2 of (x') which describe C'_n as P describes C_1. To the point of intersection of two lines of (x) [(x')] correspond in (x') [(x)] two [three] non-basic intersections of the image curves of these lines. These two [three] image points always lie at the two [three] intersections of the two curves of (x') [(x)] given by the defining equations in which the parameters are the coordinates of the common point of the two lines of (x) [(x')]. The images in both planes are distinct except for points on certain fixed curves.

If the two images P'_1, P'_2 of a point P of (x) coincide, P is on the branch-point curve of (x) which will be called $L(x)$. The locus of the corresponding coincidences is the coincidence curve of (x'), denoted by $K'(x')$, which is in (1, 1) correspondence with $L(x)$. If in (x), two of the three images of the point P' of (x') coincide, P' is on the branchpoint curve $L'(x')$ and the coincident image points $P_1 \equiv P_2$ describe the coincidence curve $K(x)$ in (x) which is in (1, 1) correspondence with $L'(x')$. At the same time the remaining image point P_3 in (x) describes a fixed locus $\Gamma(x)$, the residual image of $L'(x')$ and also in (1, 1) correspondence with it. When all three image points coincide, $K(x)$ and $\Gamma(x)$ have a common tangent at $P_1 \equiv P_2 \equiv P_3$ and P' is a cusp on $L'(x')$. There are but a finite number of points of (x') whose three image points in (x) coincide unless every point of $L'(x')$ has this property.

The equation of $L(x)$ is the condition on the x_i that the two curves of (x') given by the defining equations be tangent, $L(x)$ has no non-basic multiple points. The proper image of $L(x)$ is $K'(x')$ counted twice. $K'(x')$ may have

non-basic multiple points since it is of the same genus as $L(x)$. In like manner the equation of $L'(x')$ is the condition on the x'_i that the two curves of (x) be tangent. Aside from the basis points, the only singularities of $L'(x')$ are cusps. The proper image of $L'(x')$ is $[K(x)]^2\Gamma(x)$. Both $K(x)$ and $\Gamma(x)$ are of the same genus as $L'(x')$ and may have non-basic multiple points. When the genus of the image curves of the straight lines of one plane is known, the order of the branchpoint curve of that plane can be readily found without the above-mentioned algebraic process which is always possible, but sometimes laborious. If t is the multiplicity of the transformation and p the genus, the number of coincidences of the transformation is given by the formula $2(t+p-1)$, known as "Zeuthen's formula."* Since these coincidences are in (1, 1) correspondence with the intersections of the branchpoint curve and a general line of the other plane, the order of that branchpoint curve is $2(t+p-1)$.

All image curves of (x) have only contacts with $L(x)$ in accordance with the lemma:

If a point P of (x) describe a curve C, the necessary and sufficient condition that its images P'_1, P'_2 of (x') describe distinct curves is that C touch $L(x)$ at every common point.

This lemma was proved for (2, 2) point correspondences† where it was valid for both planes. In the case of (2, 3) point correspondences, however, it holds only for the double plane (x). For the triple plane (x') we have the lemma:

If a point P' of (x') describe a curve C', the necessary and sufficient condition that its images P_1, P_2, P_3 of (x) describe distinct curves is that C' and $L'(x')$ have contacts and intersections respectively equal to the intersections of the image of C' with $K(x)$ and $\Gamma(x)$.

Applying these lemmas to the fixed curves of (x) and (x') we have the theorem:

In (x') $L'(x')$ and $K'(x')$ have r intersections and s tangencies corresponding in (x) to r tangencies of $L(x)$ and $\Gamma(x)$ and s tangencies of $L(x)$ and $K(x)$.

If a line C'_1 meets $K'(x')$ in i' points, its image in (x) is a curve C_n tangent to $L(x)$ at i' points. The image of C_n is C'_1 counted three times, and a

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* Zeuthen, "Nouvelle démonstration de théorèmes sur des séries de points correspondants sur deux courbes," *Mat. Annal.*, Vol. III (1871), p. 150.

† F. R. Sharpe and V. Snyder, *loc. cit.*, Article 2.

residual curve that cuts C_1' in $2d+i'$ points corresponding to the d variable double points of C_n and the i' contacts of $L(x)$ and C_n. If a line C_1 meets $K(x)$ and $\Gamma(x)$ in i and j points respectively, its image in (x') is a curve C_m' which has i contacts and j intersections with $L'(x')$. The image of C_n' is C_1 counted twice, and a residual curve intersecting C_1 in $2d'+i$ points corresponding to the i contacts of C_n' and $L'(x')$ and the d' variable double points of C_n'.

4. *Types of Correspondences.*—There are twelve independent types of general (2, 3) point correspondences between two planes. Special cases may be common to two or more types. Each type is established by a set of defining equations as described in the preceding article. For convenience, these have been divided into three classes with respect to the curve system in the triple plane.

Class 1. Lines and conics; five types.

Class 2. Curves of order n having basis points of multiplicity $n-2$ at the vertex of the line pencil; four types.

Class 3. Conics with two basis points; three types.

The following table shows the curves employed in the defining equations for each type. The symbol $C_n; jP_i$ means a curve of order n with j basis points of each multiplicity i.

Class.	Type.	$u_k'(x')=0.$	$u_k(x)=0.$	$v_k'(x')=0.$	$v_k(x)=0.$
1	I	line	line	conic	cubic
	II	line	cubic	conic	line
	III	line	conic; P_1	conic	conic; P_1
	IV	line	cubic; $6P_1$	conic	cubic; $6P_1$
	V	line	conic; $2P_1$	conic	cubic; P_2P_1
2	VI	line pencil	cubic	$C_n; P_{n-2}$	line
	VII	line pencil	line pencil	$C_n; P_{n-2}$	$C_m; P_{m-3}$
	VIII	line pencil	cubic; $6P_1$	$C_n; P_{n-2}$	cubic; $6P_1$
	IX	line pencil	cubic; $8P_1$	$C_n; P_{n-2}$	$C_9; 8P_3$
3	X	conic; $2P_1$	line	conic; $2P_1$	cubic
	XI	conic; $2P_1$	conic; P_1	conic; $2P_1$	conic; P_1
	XII	conic; $2P_1$	cubic; $6P_1$	conic; $2P_1$	cubic; $6P_1$

Each of these types will now be discussed, Type I in detail and the others more briefly. Similar methods are used in investigating all the types, but the algebraic details naturally differ.

§ 2. CLASS 1. FIVE TYPES.

5. *Type I. Image of a Line.*—The defining equations,

$$\sum_{k=1}^{3} x_k x'_k = 0, \tag{1}$$

$$\sum_{k=1}^{6} u_k(x) v'_k(x') = 0, \tag{2}$$

where $u_k(x)=0$, $v'_k(x')=0$ are respectively cubics of (x) and conics of (x'), relate the point (x'_1, x'_2, x'_3) to the three intersections of the line and cubic determined by it in (x), and the point (x_1, x_2, x_3) to the two intersections of the line and conic determined by it in (x').

The image of a line, $C'_1 \equiv \sum_{k=1}^{3} a'_k x'_k = 0$, is the quintic

$$C_5 \equiv \sum_{k=1}^{6} u_k(x) v'_k(a'_2 x_3 - a'_3 x_2,\ a'_3 x_1 - a'_1 x_3,\ a'_1 x_2 - a'_2 x_1) = 0.$$

C_5 has a double point at (a'_1, a'_2, a'_3) and is of genus 5. The image of a line, $C_1 \equiv \sum_{k=1}^{3} a_k x_k = 0$, is the quintic

$$C'_5 \equiv \sum_{k=1}^{6} u_k(a_2 x'_3 - a_3 x'_2,\ a_3 x'_1 - a_1 x'_3,\ a_1 x'_2 - a_2 x'_1) v'_k(x') = 0.$$

C'_5 has a triple point at (a_1, a_2, a_3) and is of genus 3.

6. *Branchpoint and Coincidence Curves.*—The equation of $L(x)$ is the condition that the line and conic of (x') be tangent. Writing the equation of a tangent to (2) at (x'_1, x'_2, x'_3) and equating the coefficients of this equation to those of (1), we have

$$\sum_{k=1}^{6} u_k \frac{\partial v'_k}{\partial x'_i} = \rho x_i, \qquad i=1, 2, 3. \tag{3}$$

Let $\qquad v'_k \equiv a'_k x'^2_1 + b'_k x'^2_2 + c'_k x'^2_3 + 2d'_k x'_2 x'_3 + 2e'_k x'_3 x'_1 + 2f'_k x'_1 x'_2 = 0.$

Eliminating the x'_i and ρ from the three equations of (3) and from (1), we have the equation

$$L(x) \equiv \begin{vmatrix} \Sigma u_k a'_k & \Sigma u_k f'_k & \Sigma u_k e'_k & x_1 \\ \Sigma u_k f'_k & \Sigma u_k b'_k & \Sigma u_k d'_k & x_2 \\ \Sigma u_k e'_k & \Sigma u_k d'_k & \Sigma u_k c'_k & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix} = 0.$$

$L(x)$ is of order 8 and genus 21 having no singularities. The image of L_8 is K'_{20}. It is of genus 21 and has 150 double points.

The equation of $L'(x')$ is the condition on the parameters x'_i that the line and cubic of (x) be tangent. It is the equation of the cubic in line coordinates. The coefficients in the equation of the line appear to degree equal to the class

of the cubic and the coefficients of the given curve appear to degree equal to that of the discriminant of a binary cubic.* The cubic is of class 6, the discriminant of order 4, and each element in it is of degree 2 in the x'_i. Hence $L'(x')$ is of order 14. L'_{14} has thirty-nine cusps and is of genus 39. The image of L'_{14} is K_{17} counted twice and the residual curve Γ_{36}. K_{17} and Γ_{36} are each of genus 39 and have 81 and 556 double points, respectively.

The order of $L(x)$ by Zeuthen's formula is $2(2+3-1)=8$. The order of $L'(x')$ is $2(3+5-1)=14$. This formula serves as a check when the equations of the branchpoint curves and the genera of the image curves are found by other methods.

K_{17} and Γ_{36} have thirty-nine contacts corresponding to the thirty-nine cusps of L'_{14}. The three image points corresponding to a point at a cusp of L'_{14} coincide at the corresponding contact of K_{17} and Γ_{36}. L'_{14} and K'_{20} have sixty-eight contacts corresponding to the contacts of L_8 and K_{17}, and 144 intersections corresponding to the contacts of L_8 and Γ_{36}.

7. *Successive Images of Lines.*—To a line $C'_1(x')$ corresponds $C_5(x)$ having one variable double point. The image of C_5 is C'_{25} which consists of C'_1 counted three times, and a residual C'_{22} which passes through the twenty intersections of C'_1 and K'_{20} corresponding to the twenty contacts of C_5 and L_8. The other two intersections of C'_{22} and C'_1 correspond to the variable double point of C'_5. C'_{22} and L'_{14} intersect in 308 points which consist of eighty-five contacts corresponding to the eighty-five intersections of C_5 and K_{17}, and 138 intersections corresponding to 138 of the 180 intersections of C_5 and Γ_{36}. To the remaining forty-two intersections of C_5 and Γ_{36} correspond the forty-two intersections of L'_{14} and C'_1 counted three times.

To a line $C_1(x)$ corresponds $C'_5(x')$ having one variable triple point. The image of C'_5 is C_{25} consisting of C_1 counted twice and a residual C_{23}. Of the twenty-three intersections of C_1 and C_{23}, six correspond to the triple point on C'_5 and the remaining seventeen are at the seventeen intersections of K_{17} and C_1 which correspond to the seventeen contacts of C'_5 and L'_{14}. To the remaining thirty-six intersections of C'_5 and L'_{14} correspond the thirty-six intersections of Γ_{36} and C_1, which are not on C_{23}. C_{23} and L_8 intersect in 184 points which are ninety-two contacts corresponding to ninety-two of the 100 intersections of C'_5 and K'_{20}. To the remaining eight intersections of C'_5 and K'_{20} correspond the sixteen intersections of C_1 counted twice and L_8.

Two lines C'_1, $\overline{C}'_1$ of (x') have C_5, $\overline{C}_5$ for images in (x) which intersect in twenty-five points. Three of these are collinear and correspond to the common

*See Salmon's "Higher Plane Curves," 4th ed., Art. 91, 188, 222.

point of C'_1, $\overline{C}'_1$. The remaining twenty-two points have for their images the twenty-two intersections of C'_1 with $\overline{C}'_{22}$ and $\overline{C}'_1$ with C'_{22}. The images of two lines C_1, $\overline{C}_1$ of (x) are C'_5, $\overline{C}'_5$ of (x') which intersect in twenty-five points. Two of these intersections correspond to the common point of C_1, $\overline{C}_1$, and the remaining twenty-three have for their images the twenty-three intersections of C_1 with $\overline{C}_{23}$, and $\overline{C}_1$ with C_{23}, and also twenty-three intersections of C_{23} with $\overline{C}_{23}$.

8. *Cubics in* (x) *with One Basis Point.*—Let P_1 be a simple basis point on the cubics of (x). The image of C'_1 is C_5 passing through P_1 and having a variable double point. The image of C_1 is C'_5 which has a variable triple point. The image of P_1 is the fixed line f'_1. The image of f'_1 is P_1 and a residual f_5 with a triple point at P_1. K_{17} and f_5 have the same three tangents at P_1.

$L(x)$ is of order 8 and genus 20, having a double point at P_1. Its proper image is K'_{19}. $L'(x')$ is of order 14 and genus 39 having thirty-nine cusps. K_{17} has a triple point and Γ_{36} an eight-fold point at P_1.

A line C_1 through P_1 corresponds to f'_1 and C'_4. C'_4 has a variable double point on f'_1 and is of genus 2. The two simple intersections of f'_1 and C'_4 are images of the direction of C_1 through P_1. The image of C'_4 is C_1 counted twice and a residual C_{18} with a double point at P_1. C_{18} meets C_1 in sixteen points apart from P_1, fourteen of which are at the non-basic intersections of C_1 and K_{17}, and two correspond to the variable double point of C'_4.

9. *Conics in* (x) *with One Basis Point.*—Let the simple basis point be P'_1. The image of C'_1 is C_5 which has a variable double point. The image of C_1 is C'_5 passing through P'_1 and with a variable triple point. The image of P'_1 is the fundamental line f_1. To f_1 corresponds P'_1 and a residual f'_5 with a four-fold point at P'_1. K'_{20} and f'_5 have the same four tangents at P'_1. $L(x)$ is of order 8 and genus 21. K'_{20} has a four-fold point at P'_1 and 144 double points. $L'(x')$ is of order 14 with a four-fold point at P'_1. It is of genus 36 and has thirty-six cusps. K_{16} and Γ_{34} have 66 and 487 non-basic double points, respectively.

To a line C'_1 through P'_1 corresponds f_1 and a C_4 of genus 3. Two of the intersections of f_1 and C_4 correspond to the direction of C'_1 through P'_1. The image of C_4 is C'_1 counted three times and a residual C'_{17} passing through P'_1 simply. C'_{17} meets C'_1 in sixteen points apart from P'_1 which are at the sixteen non-basic intersections of K'_{20} and C'_1.

10. *Cubics in* (x) *with* $i \leqq 9$ *Basis Points and Conics in* (x') *with* $j \leqq 4$ *Basis Points.*—In general there may be any combination of i simple basis points, $i \leqq 9$, on the cubics of (x), and $j \leqq 4$ on the conics of (x'), and any par-

ticular case may be obtained from the following results by giving to i and j their appropriate values. Let the basis points of (x) be denoted by P_i and those of (x') by P'_j. The image of C'_1 is C_5 passing simply through each of the P_i of genus 5 and having a variable double point. The image of C_1 is C'_5 passing simply through each of the P'_j of genus 3 and having a variable triple point. To each basis point P_i corresponds a fixed line f'_i. To each f'_i corresponds the P_i whose image it is, and a residual f_5 with a triple point at that P_i and passing simply through the remaining $i-1\,P_i$. To each P'_j corresponds a fixed line f_j. The image of f_j is the P'_j to which it corresponds and a residual f'_5 having a four-fold point at that P'_j and passing simply through the the remaining $j-1\,P'_j$.

$L(x)$ is of order 8 and genus $21-i$, having a double point at each of the P_i. K'_{20-i} has each of the P'_j of multiplicity 4 and $\frac{1}{2}(i-20)(i-15)-12j$ non-basic double points. L'_{14} is of genus $39-3j$ having $39-3j$ cusps and a four-fold point at each of the P'_j. K_{17-j} has a triple point at each of the P_i and $\frac{1}{2}(j^2-25j-6i+162)$ double points. Γ_{36-2j} has an eight-fold point at each of the P_i and $2(j^2-33j-14i+278)$ double points. Apart from the P'_j, L'_{14} and K'_{20-i} have $68-3i-4j$ contacts and $144-8i-8j$ intersections corresponding to the non-basic contacts of L_8 with K_{17-j} and Γ_{36-2j}, respectively.

The images of a general line through one basis point of either plane are similar to those in the cases previously discussed. The lines joining two or more basis points will now be considered. Since the cubics and conics are not composite, there can be no more than three collinear basis points in (x) and two in (x'). The image of the line joining two basis points P_1, P_2 of (x) consists of f'_1, f'_2 and a fixed C'_3 through their intersection and through all the P'_j. Let three basis points P_1, P_2, P_3 be collinear. To the line joining them corresponds f'_1, f'_2, f'_3 all concurrent and a fixed C'_2 not through their common intersection, but through the P'_j. The image of the line joining two basis points P'_1, P'_2 of (x') consists of f_1, f_2 and a fixed C_3 not through their intersection, but through all the P_i.

11. *Line Pencil in Each Plane.*—Let the lines in (x) given by equation (1) form a pencil whose vertex $P\equiv(0,0,1)$ is not on the cubics. Then the lines of (x') also form a pencil whose vertex $P'\equiv(0,0,1)$ can not be on the conics. The image of C'_1 is C_5 with a double point at P and of genus 5. To C_1 corresponds C'_5 with a triple point at P' and of genus 3. The image of P' is the cubic f_3 whose image in (x') is P' and a residual f'_{15} with a twelve-fold point at P'. The image of P is the conic f'_2 to which corresponds P and a residual f_{10} with a six-fold point at P. To the line p' through P' corresponds

f_8 and the line p counted twice through P. To p corresponds f'_2 and p' counted three times. The rays of the two pencils are in (1, 1) correspondence. $L(x)$ is of order 8 and genus 20 having a double point at P. K'_{18} has a twelve-fold point at P' and fifty double points. L'_{14} has thirty cusps, a six-fold point at P' and is of genus 33. K_{14} has a six-fold point at P and thirty double points. Γ_{24} has a sixteen-fold point at P and 100 double points.

12. Now assume that both defining equations are satisfied by $x_2=0$, $x'_1=0$. The image of C'_1 is $x_2=0$ and C_4 of genus 3 through P. The image of C_1 is $x'_1=0$ and C'_4 with a double point at P' and of genus 2. To P' corresponds $x_2=0$ and a conic f_2 not through P. To P corresponds $x'_1=0$ and the line f'_1 not through P'. The image of $x_2=0$ is $x'^3_1=0$ and f'_1. The image of $x'_1=0$ is $x^2_2=0$ and f_2. L_8 has $x^2_2=0$ as a component, the proper curve being a sextic of genus 10 and not through P. K'_{18} consists of $x'^6_1=0$ and a proper C'_{12} with a six-fold point at P'. L'_{14} has $x'^4_1=0$ as a component and a proper C'_{10} with a double point at P', of genus 17 and having eighteen cusps. K_{14} consists of $x^4_2=0$ and a C_{10} with a double point at P. Γ_{24} consists of $x^8_2=0$ and a C_{16} with a six-fold point at P. The fixed components $x_2=0$, $x'_1=0$ are illustrations of the curves D_1, D_2 discussed by R. Baldus.*

13. When, in addition to the line pencils, there are $i\leqq 7$ basis points on the cubics of (x) and $j\leqq 3$ basis points of the conics of (x'), the image of C'_1 is C_6 with a double point at P and simple points at each of the P_i, and the image of C_1 is C'_5 with a triple point at P', and passing simply through each of the P'_j. L_8 has double points at P and at each of the P_i. K'_{18-i} has a $(12-i)$-fold point at P and triple points at each of the P'_j. L'_{14} has a six-fold point at P' and four-fold points at each of the P'_j. K_{14-j} has a $(6-j)$-fold point at P and double points at each of the P_i. Γ_{24-2j} has a $(16-2j)$-fold point at P and four-fold points at each of the P_i. The image of P' is f_3 through the P_i, but not through P. The image of P is f'_2 through the P'_j, but not through P'. To each P'_j corresponds a line through P, and to each P_i a line through P'.

14. *Geometric Depiction of Pencil Cases.*—All the cases in which the bilinear equation has but two terms can be visualized after the method of Marletta † by means of a quintic surface in ordinary space. Let $F=0$ be a surface of order 5 with a triple point at P_3 and a double point at P_2. Choose two planes (x) and (x') not through P_2 or P_3 as the double and triple plane, respectively, upon which the correspondence is to be pictured. Take any point P in (x) and any point P' in (x'). Connect P with P_3. The line PP_3 meets F in

* Baldus, *loc. cit.*, 2, Articles 6, 7, 8.

† G. Marletta, *loc. cit.*

two points not at P_3. Project these two points from P_2 upon (x') obtaining in (x') two points that are the images of the point P in (x). Going the other way, join P' with P_2 and project the three points of intersection of $P'P_2$ and F, not at P_2, on (x) from P_3. This gives in (x) the three images of P'.

The apparent contour of F from P_2, *i. e.*, the tangent cone, is cut by (x') in the curve L'_{14}. Every tangent line, elements of the tangent cone from P_2, meets F in one other point. The projections of the locus of the points of tangency and the locus of the residual intersections upon (x) from P_3 give the curves K_{14} and Γ_{24}, respectively. In like manner the tangent cone to F from P_3 is cut by (x) in L_8, and its curve of contact is projected upon (x') from P_2 into K'_{18}.

If $x_2=0$, $x'_1=0$ satisfy both defining equations, the surface F_5 consists of a plane through P_2P_3 and a residual surface F_4 passing through P_2 and having a double point at P_3. The depiction on (x) and (x') may now be obtained as before. Basis points in (x) and (x') are accounted for by fixed lines on F passing through P_2 and P_3 respectively.

15. *Type II.*—The defining equations of this type are obtained by interchanging the parameters of the line and conic of (x'). The image of a line, $C'_1\equiv\sum\limits_{k=1}^{3} a'_k x'_k=0$, is,

$$C_7\equiv\sum_{k=1}^{3} x_k v'_k(a'_2u_3-a'_3u_2,\ a'_3u_1-a'_1u_3,\ a'_1u_2-a'_2u_1)=0.$$

Since the three cubics

$$u_1/a'_1=u_2/a'_2=u_3/a'_3$$

form a pencil, their nine intersections are variable double points on C_7, leaving it of genus 6. The image of C_1 is C'_7 with four variable triple points at the common points of the pencil of conics whose parameters are those of C_1. C'_7 is of genus 3.

The equation of $L(x)$ is found as in Type I. L_8 is of genus 21 and has no singularities. K'_{28} is of genus 21 and has 330 double points. As in Type I we find that L'_{16} has fifty-seven cusps and is of genus 48. K_{23} and Γ_{66} have 183 and 2032 double points respectively.

16. *Type III.*—The defining equations are

$$\sum_{k=1}^{5} u_k(x)v'_k(x')=0, \tag{1}$$

$$\sum_{k=1}^{3} v_k(x)x'_k=0, \tag{2}$$

wherein $u_k(x)=0$, $v_k(x)=0$ are conics of (x) having a basis point at $R\equiv(1,0,0)$ and $v'_k(x')=0$ are general conics of (x'). To C'_1 corresponds C_6 with a triple point at R and three variable double points at the other three intersections of a pencil of conics whose parameters are those of C'_1. C_6 is of genus 4. The image of C_1 is C'_6 with seven variable double points and of genus 3. If C_1 passes through R its proper image is a cubic of genus 1. The image of of R is a fixed cubic of genus 1.

L_8 has a four-fold point at R and is of genus 15. K'_{18} has 121 double points. The equation of $L'(x')$ is the condition that two conics through a fixed point have a contact not at that point. This condition is of order 4 in the coefficients of each of the conics. Then $L'(x')$ is of order 12. It has twenty-seven cusps and is of genus 28. K_{20} has R of multiplicity 11, and eighty-eight double points. Γ_{32} has a fourteen-fold point at R and 346 double points.

17. *Type IV*.—The correspondence is defined by equations of the same form as those in Type III, wherein now $u_k(x)=0$, $v_k(x)=0$ are cubics through six simple basis points P_i of (x). The image of C'_1 is C_9 with each of the P_i three-fold, and with double points at the three variable intersections of a pencil of cubics whose parameters vary with C'_1. To C_1 corresponds C'_9 with twenty-three variable double points. If C_1 passes through one P_i its proper image is C'_6. To each P_i corresponds a fundamental cubic. If C_1 contains two P_i its proper image is a fixed C'_3. The proper image of a cubic through the six P_i is a C'_9.

L_{12} has a four-fold point at each P_i and is of genus 19. K'_{18} has 117 double points. The equation of $L'(x')$ is the condition that the two cubics through the P_i be tangent. This condition is of order 6 in the coefficients of each. L'_{18} has seventy-five cusps and is of genus 61. K_{33} has an eleven-fold point at each P_i and 105 double points. Γ_{96} has a thirty-two-fold point at each P_i and 1428 double points.

18. *Type V*.—The defining equations are written as in Type III. For the present type $u_k(x)=0$ represent cubics with a double point at $P\equiv(0,0,1)$ and a simple point at $R\equiv(1,0,0)$ and $v_k(x)=0$ are conics through P and R. The image of C'_1 is C_7 with P four-fold, R three-fold and two double points at the two variable intersections of a pencil of conics through P and R whose parameters vary with C'_1. C_7 is of genus 4.

In order to study the properties of the image of C_1, we may write the defining equations as follows:

$$\sum_{k=1}^{6} u_k(x)v'_k(x') \equiv x_2(a'x_1^2+b'x_2x_3+c'x_1x_3+e'x_1x_2+f'x_2^2)+d'x_1^2x_3=0, \tag{1}$$

$$\sum_{k=1}^{3} v_k(x)x'_k \qquad \equiv x_2(p'x_1+q'x_2+r'x_3)+s'x_1x_3=0, \tag{2}$$

wherein a', b', c', d', e', f' are quadratic and p', q', r', s' linear in the x'_i. Multiply (1) by s', (2) by $d'x_1$ and subtract, using the resulting equation

$$s'(a'x_1^2+b'x_2x_3+c'x_1x_3+e'x_1x_2+f'x_2^2)-d'x_1(p'x_1+q'x_2+r'x_3)=0, \tag{3}$$

with (2) as the pair of defining equations. Eliminate the x_i from the equation of C_1 and (2) and (3) and we obtain a C'_8 with double points at the two intersections of $s'=0$, $d'=0$ and containing $s'=0$ as a fixed component. Then the proper image of C_1 is C'_7, which passes simply through the two basis points $s'=0$, $d'=0$. It is of genus 4 and has eleven variable double points.

The proper image of a line through P is a cubic which does not pass through $s'=0$, $d'=0$. The image of P is f'_4. The proper image of a line through R is a quartic not through $s'=0$, $d'=0$. To R corresponds f'_3. Both f'_4 and f'_3 pass through $s'=0$, $d'=0$. The image of the fundamental line PR is $f'_3f'_4$. The image curves of (x) intersect PR only at the basis points P and R. Every point of PR except P and R corresponds to the two basis points $s'=0$, $d'=0$. To each of the basis points $s'=0$, $d'=0$ corresponds the line PR. The image of $s'=0$ is PR counted twice, and a fixed C_5 with a triple point at P and a double point at R. The image of $d'=0$ is PR counted twice and a fixed C_{12}, with P seven-fold and R five-fold.

L_{10} has P six-fold and R four-fold and is of genus 15. K'_{17} has 105 double points. The condition that the cubic and conic of (x) defined by (1) and (2) be tangent is of order 4 in the coefficients of each, giving $L'(x')$ of order 12. L'_{12} has twenty-seven cusps and is of genus 28. Neither L'_{12} nor K'_{17} pass through the basis points $s'=0$, $d'=0$. K_{23} has a fourteen-fold point at P, a nine-fold point at R and seventy-six double points. Γ_{39} has a twenty-fold point at P, an eighteen-fold point at R and 295 double points.

§ 3. Class 2. Four Types.

19. *Type VI.*—The defining equations are

$$\sum_{k=1}^{3} x_k\psi'_k(x')=0, \tag{1}$$

$$x'_1u_1(x)+x'_3u_3(x)=0, \tag{2}$$

wherein $u_1(x)=0$, $u_3(x)=0$ are cubics of (x), and

$$\psi'_k(x') \equiv x_2'^2u'_k(x'_1, x'_3)+x'_2v'_k(x'_1, x'_3)+w'_k(x'_1, x'_3)=0,$$

(u'_k, v'_k, w'_k being homogeneous functions of x'_1, x'_3 of the respective degrees $n-2$, $n-1$, n) are curves of order n with $(n-2)$-fold points at $Q' \equiv (0,1,0)$, the vertex of the line pencil of (x'). The cubics of (x) would have no basis points but for the fact that the lines of (x') form a pencil. This forces the cubics of (x) to form a pencil, thus introducing nine simple basis points P_i into (x).

The image of C'_1 is C_{3n+1} with an n-fold point at each P_i and of genus $3n$. If C'_1 passes through Q' its proper image is a cubic counted twice through the P_i. The image of Q' is f_{3n-5} with an $(n-2)$-fold point at each P_i. To C_1 corresponds C'_{3n+1} with a $(3n-5)$-fold point at Q', and a triple point at each of the $4n-4$ variable intersections of a pencil of curves of order n whose parameters vary with C_1. C'_{3n+1} is of genus $3n-3$. If C_1 passes through a P_i, its proper image is C'_{2n+1} with Q' of multiplicity $2n-3$. If C_1 passes through two P_i its proper image is C'_{n+1} with an $(n-1)$-fold point at Q'. The image of each P_i is f'_i of order n with Q' $(n-2)$-fold. The proper image of a cubic through the P_i is a line counted three times through Q'. The line pencil of (x') is in $(1,1)$ correspondence with the pencil of cubics of (x).

To find the equation of $L(x)$, solve equation (2) for x'_3 in terms of x'_1, substitute this value of x'_3 in (1) and rearrange, obtaining a quadratic in x'_2/x'_3 whose discriminant equated to zero is $L(x)$. $L(x)$ is of order $6n-4$ and genus $12n-12$ with a $(2n-2)$-fold point at each P_i. $K'(x')$ is of order $6n-2$ with a $(6n-8)$-fold point at Q' and $18n-18$ double points. $L'(x')$, found as in Type I, is of order $6n+4$, genus $30n-15$, has Q' $(6n-8)$-fold and $36n-18$ cusps. $K(x)$ is of order $12n-12$ and has a $(4n-2)$-fold point at each P_i and $18n-6$ double points. $\Gamma(x)$ is of order $48n-32$ and has a $(16n-12)$-fold point at each P_i and $162n-126$ double points. Apart from Q', L'_{6n-4} and K'_{6n-2} have $24n-14$ contacts and $60n-44$ intersections corresponding to the non-basic contacts of L_{6n-4} with K_{12n-2} and Γ_{48n-32} respectively.

20. *Type VII.*—The defining equations are

$$\sum_{k=1}^{t} \phi_k(x)\psi'_k(x') = 0, \tag{1}$$

$$x_1x'_1 + x_2x'_3 = 0. \tag{2}$$

$$\phi_k(x) \equiv x_3^3 u_k(x_1, x_2) + x_3^2 v_k(x_1, x_2) + x_3 w_k(x_1, x_2) + s_k(x_1, x_2) = 0$$

(wherein u_k, v_k, w_k, s_k are homogeneous functions of the respective degrees $m-3$, $m-2$, $m-1$, m in x_1, x_2) is a C_m with an $(m-3)$-fold point at $P \equiv (0,0,1)$, the vertex of the line pencil of (x). $\psi'_k(x')$ is defined as in Type VI. Equation (1) has t terms where t takes the value $3n$ or $4m-2$

according as m is greater or less than $1/4(3n+2)$. A special case of this type for $n=2$, $m=3$ has been discussed in Article 11.

The image of C'_1 is C_{m+n} with an $(m+n-3)$-fold point at P and of genus $2(m+n)-5$. The image of C_1 is C'_{m+n} with an $(m+n-2)$-fold point at Q' and of genus $m+n-2$. The proper image of a line p through P is a line q' counted three times through Q', and the proper image of q' is p counted twice. The pencils of (x) and (x') are in $(1, 1)$ correspondence. To P corresponds f'_{m+n-3} with Q' $(m+n-5)$-fold. To Q' corresponds f_{m+n-2} with an $(m+n-5)$ fold point at P.

$L(x)$, obtained as in Type VI, is of order $2(m+n)-2$ and genus $10(m+n)-30$, having a $[2(m+n)-8]$-fold point at P. $K'(x')$ is of order $6(m+n)-12$ having Q' of multiplicity $6(m+n)-18$ and $20(m+n)-50$ double points. To obtain the equation of $L'(x')$, solve equation (2) for x_2 in terms of x_1 and substitute this value of x_2 in (1). Then (1) may be written as a cubic in x_3/x_1 whose discriminant equated to zero is the equation of $L'(x')$. It is of order $4(m+n)-6$ and genus $16(m+n)-47$ with Q' of multiplicity $4(m+n)-14$ and $12(m+n)-30$ cusps. $K(x)$ is of order $4(m+n)-6$ with P of multiplicity $4(m+n)-14$ and $12(m+n)-30$ double points. $\Gamma(x)$ is of order $8(m+n)-16$ with P of multiplicity $8(m+n)-24$ and $40(m+n)-100$ double points.

21. *Notation.*—In the types of Classes 2 and 3 the defining equations represent respectively the same curve systems in the double plane as in Class 1. Thus the discussion of the succeeding types is so similar to the corresponding types of the first class that only a tabulation of the results is essential in most cases. The following notation will be used in this tabulation:

The symbol "$\sim$" meaning "corresponds to";

L, K', L', K, Γ, fixed curves as defined heretofore;

f, f', fundamental curves of (x) and (x');

C, C', variable curves of (x) and (x');

P, Q, R, P', Q', R', basis points of (x) and (x');

$\overline{P}$, non-basic but fixed points of either plane;

$\overline{\overline{P}}$, variable points of either plane;

p, genus of curve being described;

k, cusps of L';

Subscripts of curves denote their order;

Subscripts of points denote their multiplicity on the curve being described; (the subscripts i and j, however, denote any one of a given number of basis points or curves playing the same rôle).

The following types will illustrate the use of this notation:

22. *Type VIII.*—The defining equations are

$$\sum_{k=1}^{4} u_k(x)\,\psi'_k(x') = 0, \tag{1}$$

$$x'_1 v_1(x) + x'_3 v_3(x) = 0. \tag{2}$$

in which $u_k(x)=0$, $v_1(x)=0$, $v_3(x)=0$, are cubics with six simple basis points P_i. The cubics of equation (2) form a pencil because their parameters belong to a line pencil, and thus introduce three additional basis points Q_j into (x). The pencil of cubics of (x) and the line pencil of (x') are in (1, 1) correspondence.

$C'_1 \sim C_{3n+3}$, $p=3n+1; 6P_{n+1}, 3Q_n$.
$C_1 \sim C'_{3n+3}$, $p=3n-1; Q'_{3n-3}, (12n-4)\overline{\overline{P}}_2$.
$P_i \sim f'_{i,\,n+1}; Q'_{n-1}$.
$Q_j \sim f'_{j,\,n}; Q'_{n-2}$.
$Q' \sim f_{3n-3}; 6P_{n-1}, 3Q_{n-2}$.
L_{6n}, $p=12n-8; 6P_{2n}, 3Q_{2n-2}$.
K'_{6n}, $p=12n-8; Q'_{6n-6}, (18n-12)\overline{P}_2$.
L'_{6n+6}, $p=30n-15; Q'_{6n-6}, (36n-6)k$.
K_{12n+6}, $p=30n-15; 6P_{4n+2}, 3Q_{4n-2}, 18n\overline{P}_2$.
Γ_{48n-12}, $p=30n-15; 6P_{16n-4}, 3Q_{16n-8}, (167n-72)\overline{P}_2$.

23. *Type IX.*—The defining equations are written as in Type VIII. For the present type, $v_1(x)=0$, $v_3(x)=0$ represent cubics of (x) passing through eight basis points P_i and therefore determining a ninth Q; $u_k(x)=0$ represents curves of order 9 with triple points at each of the eight P_i, but not passing through Q.

$C'_1 \sim C_{3n+9}$, $p=3n+4; 8P_{n+3}, Q_n$.
$C_1 \sim C'_{3n+9}$, $p=3n+5; Q'_{3n+3}, (12n+20)\overline{\overline{P}}_2$.
$Q' \sim f_{3n+3}; 8P_{n+1}, Q_{n-2}$.
$P_i \sim f'_{i,\,n+3}; Q'_{n+1}$.
$Q \sim f'_n; Q'_{n-2}$.
L_{6n+12}, $p=12n+4; 8P_{2n+4}, Q_{2n-2}$.
K'_{6n+6}, $p=12n+4; Q'_{6n}, (18n+6)\overline{P}_2$.
L'_{6n+12}, $p=30n+25; Q'_{6n}, (36n+30)k$,
K_{12n+30}, $p=30n+25; 8P_{4n+10}, Q_{4n-2}, (18n+18)\overline{P}_2$.
Γ_{48n+48}, $p=30n+25; 8P_{16n+16}, Q_{16n+4}, (162n+90)\overline{P}_2$.

§ 4. Class 3. Three Types.

24. *Type X.*—The defining equations are

$$\sum_{k=1}^{4} u_k(x)\,c_k'(x')=0, \tag{1}$$

$$\sum_{k=1}^{3} x_k v_k'(x')=0, \tag{2}$$

wherein $c_k'(x')=0$, $v_k'(x')=0$ are conics of (x') with basis points $Q'\equiv(0,1,0)$ and $R'\equiv(1,0,0)$ and $u_k(x)=0$ are general cubics of (x). The image of C_1 is C_8' with four-fold points at Q' and R' and triple points at the two variable intersections of a pencil of conics whose parameters depend on those of C_1. C_8' is of genus 3. To obtain the image of C_1' we may write the defining equations as follows:

$$ax_3'^2+bx_2'x_3'+cx_1'x_3'+dx_1'x_2'=0, \tag{1}$$

$$px_3'^2+qx_2'x_3'+rx_1'x_3'+sx_1'x_2'=0, \tag{2}$$

wherein a, b, c, d and p, q, r, s are respectively general cubic and linear functions of x_1, x_2, x_3. Multiply (1) by s, (2) by d, subtract (2) from (1), remove the common factor x_3 and we obtain the equation

$$(cs-rd)x_1'+(bs-qd)x_2'+(as-pd)x_3'=0. \tag{3}$$

Using (3) with either (1) or (2) as defining equations, the image of C_1' is found to be C_9 with double points at the sixteen intersections of a pencil of quartics whose parameters vary with C_1'. Three of these double points lie at the intersections of $s=0$, $d=0$, and the remaining thirteen are variable. The line $s=0$ is a fixed component of C_9 so the proper image of C_1' is C_8 passing through the three basis points $s=0$, $d=0$ and having thirteen variable double points. C_8 is of genus 8.

L_8, $p=21$.
K_{32}', $p=21$; Q_{16}', R_{16}', $(204)\overline{P}_2$.
L_{20}', $p=39$; Q_{10}', R_{10}', $42k$.
K_{18}, $p=39$; $97\overline{P}_2$.
Γ_{44}, $p=39$; $864\overline{P}_2$.

25. The following statements hold for all three types of Class 3. The images of Q' and R' are respectively the basis curves

$$bs-dq=0, \quad cs-rd=0.$$

The line $Q'R'$ is the image of each of the three basis points $s=0$, $d=0$ of (x). All the curves of (x') intersect $Q'R'$ only in points Q' and R' which are always points of equal multiplicity. There is no proper image of the line $Q'R'$, but

every point of it except Q' and R' corresponds to the three basis points $s=0$, $d=0$. The equation of $L(x)$ is the condition that the two conics of (x') be tangent, which is of order 2 in the coefficients of each. The curves $L(x)$, $K(x)$, $\Gamma(x)$ do not pass through the basis points $s=0$, $d=0$.

26. *Type XI.*—The defining equations are

$$\sum_{k=1}^{4} u_k(x)\, c'_k(x')=0, \tag{1}$$

$$\sum_{k=1}^{4} v_k(x)\, v'_k(x')=0, \tag{2}$$

wherein $u_k(x)=0$, $v_k(x)=0$ are conics with one basis point P.

$C'_1\sim C_8,\ p=6;P_4,\ (s=0,\ d=0)_1,\ 9\overline{\overline{P}}_2.$
$C_1\sim C'_8,\ p=3;Q'_4,\ R'_4,\ 6\overline{\overline{P}}_2.$
$Q'\sim f_4;P_2,\ (s=0,\ d=0)_1.$
$R'\sim f_4;P_2,\ (s=0,\ d=0)_1.$
$P\sim f'_4;Q'_2,\ R'_2.$
$L_8,\ p=15;P_4.$
$K'_{24},\ p=15;Q'_{12},\ R'_{12},\ 106\overline{P}_2.$
$L'_{16},\ p=25;Q'_8,\ R'_8,\ 24k.$
$K_{18},\ p=25;P_{10},\ 66\overline{P}_2.$
$\Gamma_{28},\ p=25;P_{12},\ 260\overline{P}_2.$

27. *Type XII.*—In the defining equations, $u_k(x)=0$, $v_k(x)=0$ are cubics with six simple basis points P_i.

$C'_1\sim C_{12},\ p=10;6P_4,\ (s=0,\ d=0)_1,\ 9\overline{\overline{P}}_2.$
$C_1\sim C'_{12},\ p=5;Q'_6,\ R'_6,\ 20\overline{\overline{P}}_2.$
$P_i\sim f'_{i,4};Q'_2,\ R'_2.$
$Q'\sim f_6;6P_2,\ (s=0,\ d=0)_1.$
$R'\sim f_6;6P_2,\ (s=0,\ d=0)_1.$
$L_{12},\ p=19;6P_4.$
$K'_{24},\ p=19;Q'_{12},\ R'_{12},\ 102\overline{P}_2.$
$L'_{24},\ p=55;Q'_{12},\ R'_{12},\ 66k.$
$K_{30},\ p=55;6P_{10},\ 81\overline{P}_2.$
$\Gamma_{84},\ p=55;6P_{28},\ 1080\overline{P}_2.$

§ 5. COMPLETENESS OF THE CLASSIFICATION.

28. *Reduction of Certain Cases to Type Form.*—It remains now to be proved that all cases of general (2, 3) point correspondences are birationally equivalent to certain of these twelve independent types. All cases where the

defining equations are made up of combinations of the curve systems used in the twelve types (except those that define (2, 3) compound involutions) either belong to these types or can be reduced to certain of them by quadratic transformations. These curve systems are as follows: In (x') lines and conics; line pencil and curves of order n with the vertex of the pencil $(n-2)$-fold; conics with two basis points. In (x) lines and cubics; line pencil and curves of order m with the vertex of the pencil $(m-3)$-fold; conics with one basis point; cubics with six basis points; cubics with a double and a simple basis point through which pass conics; cubics with eight basis points and curves of order 9 with triple points at each of them.

In Class 1, all the cases in which the lines of (x') form a pencil are particular cases of the corresponding types of Class 2 for $n=2$. When the conics of (x') form a pencil, they may be transformed into a line pencil by quadric inversion, by which also the lines of (x') are transformed into conics with three basis points. These cases are then birationally equivalent to particular cases of Class 2. All the above cases would be independent types of Class 1 if they were not forced, as it were, into special cases of types of Class 2 by the fact that the lines or conics of (x') can have but two homogeneous parameters because these parameters represent a pencil in (x). If in Type V the parameters of the line and conic are interchanged, the cubics of (x), since their parameters are now lines of (x'), can have but three homogeneous parameters and therefore have two basis points besides those required by the system in (x). By quadric inversion these cubics go into conics through the double point and the two added basis points. The conics of (x) through the two required basis points are invariant. Then the conics, transforms of the cubics, and the invariant conics have one point in common, so this case reduces to a special case of Type III.

In Class 2 if the system in (x) consists of lines and cubics and the lines are forced into a pencil by having the line pencil of (x') for parameters, we have a special case of Type VII for $m=3$. When the system of (x) is composed of conics with a common basis point, the conics having the line pencil of (x') as parameters form a pencil which goes into a pencil of lines by quadric inversion. By the same process the conics in the other equation are transformed into cubics none of whose basis points need be at the vertex of the line pencil. This case, then, is birationally equivalent to Type VII for $m=3$. When the system in (x) consists of cubics with a double and a simple point and conics through these points, if the conics form a pencil they correspond to a line pencil; the cubics are invariant under quadric inversion

so this also reduces to a special case of Type VII. When the parameters are interchanged the cubic pencil, since it contains a double point is transformed into a pencil of conics and thence into a pencil of lines. The conics given by the other equation are transformed into cubics with three basis points, none of which are at the vertex of the line pencil. This is also a special case of Type VII.

In Class 3 when the conics of (x') form a pencil in one equation we have a particular case of the type of Class 2 having the same system of curves in (x). When the system in (x) consists of cubics with a double and a simple point and conics through them, the cubics are given two more basis points because the conics of (x') allow but four homogeneous parameters in the equation. The cubics then reduce to conics through two points one of which is the double point, and as the conics given by the other equation may remain invariant, this case is birationally equivalent to a special case of Type IX.

If, in any combination of the curve systems used in the twelve types, the associated parameters so restrict each other that the curves of each of the two defining equations become pencils in their respective planes, a (2, 3) compound involution is established.

29. *Reduction of All Other Cases.*—Since any combination of the foregoing curve systems of (x) and (x') that has at least three parameters in one of its defining equations is birationally equivalent to some one of the twelve independent types, in order to prove the classification complete it remains only to show that any curve system having two [three] variable intersections is birationally equivalent to some one of the above curve system of (x') $[(x)]$. The proof of this is the same as that for the curve systems of the simple planes of the (1, 2) and (1, 3) point correspondences. From the inequalities connecting intersections of curves of such systems, it follows by reasoning exactly like that used by Bertini* that any given non-involutorial curve system intersecting in two [three] points can be reduced by a series of quadratic transformations to one of the preceding systems of (x') $[(x)]$. Any algebraic method of establishing a general (2, 3) point correspondence between two planes is therefore birationally equivalent to some one of the given twelve independent types.

CORNELL UNIVERSITY, 1917.

* E. Bertini, "Ricerche sulle trasformazioni univoche involutorie nel piano," *Annali di Matematica*, Ser. 2, Vol. VIII (1877), pp. 244—286.

The Classification of Plane Involutions of Order (3).

By Anna Mayme Howe.

Introduction and General Discussion.

1. The purpose of this paper is to discuss all the different algebraic (1, 3) point correspondences between two planes. Such a correspondence is established between two planes (x) and (y) when a relation exists such that to any point in (x) corresponds a point (y), but to any point in (y) corresponds three points (x).

The images of lines on either plane are curves of order m in the other. The simple plane (x) contains curves which are nets, images of lines in (y).

These curves have n variable intersections which are the n points of (x) corresponding to a point (y) determined by the intersection of two lines whose images are the two curves; the remaining $m-n$ intersections are fixed and are common to all the image curves of the net. If two of the n points coincide, two curves of the net are tangent at the point. The locus of all these points of tangency is the curve of coincidences. We shall designate it by K. It is always a component of the Jacobian of the net.

There exists in the (y) plane a curve of branch points whose points are in (1, 1) correspondence with those of K. This curve of branch points, denoted by L, is a fixed curve, every point of which has at least two of its n image points coinciding in a definite direction at a point on the coincidence curve. In case $m>2$ the remaining non-coincident image points describe another locus, the residual image of L. This locus will be denoted by J'. The points at which more than two of the n images of a point in (y) coincide are intersections of the residual image with the curve of coincidences.

The first (1, 2) transformations discussed were those by Geiser.*

In this discussion a cubic surface is projected on a double plane from a point on it, and also mapped on a single plane by means of two non-intersecting lines on the surface.

In 1871 Clebsch* devised a (1, 2) point correspondence between two planes depending on the bisection of Abelian functions connected with plane

* "Ueber zwei geometresche Probleme," *Crelle's Journal für die Mathematik*, Vol. LXVII (1867), pp. 78–89.

conics, the plane of the conic being so related to a surface that to a point on the surface corresponds one point in the plane, but to a point in the plane corresponds two points on the surface.

The next step was made by de Paolis † who worked out the general theory of (1, 2) point correspondence between two planes as a generalization of the rational transformation of Cremona.

This is the first published treatment of multiple correspondence by means of an algebraic method.

The classification of all possible birational (involutorial) transformations associated with the (1, 2) types was completed by Bertini. ‡

Also F. M. Morgan. § In 1884 Chizzoni ‖ developed the general theory of (1, n) point correspondence by a plane involution of order n in which he considered a given plane as containing ∞^2 series of groups of n points such that each group is fully determined by any one of the points.

Other writers that have made noteworthy contributions to the theory of (1, n) point correspondence between two planes are G. Castelnuovo,¶ Miss Charlotte A. Scott,** and Amerigo Bottari.††

All of these treat the subject for a general n except Miss Scott who, in addition, works out several examples for special values of n greater than 2.

Enumeration of Independent Types.

2. In the theory of (1, 3) point correspondence between two planes there are five types, no one of which can be derived from another by means of a

* "Ueber den Zusammenhang einer Classe von Flächenabbildungen mit der Zweitheilung der Abel'schen Functionen," *Mathematische Annalen*, Vol. III (1871), pp. 45–75.

† (1) "Le transformazioni doppia."

(2) "La transformazione piano doppia di secondo ordine, e le suo applicazione alla geometria non euclidea."

(3) "La transformazione piano doppia di terzo ordine prime genere, e la sua applicazione a curve del quarto ordine," *Atti della R. Accademia dei Lincei*, Series 3, Vol. XII (1877).

‡ "Ricerche sulle transformazioni univoche involutorie nel piano," *Annali di Matematica*, Series 2, Vol. VIII (1877), pp. 244–286.

§ "Involutorial Transformations," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXV (1913), pp. 79–104.

‖ "Sopra le involuzioni piani," *Atti della R. Accademia dei Lencei*, Series 3, T. 19 (1884), pp. 301–371.

¶ "Sulla razionalita delle involuzioni piane," *Mathematische Annalen*, Vol. XLIV (1894), pp. 125–155.

** "Studies in Transformation of Plane Algebraic Curves," *Quarterly Journal of Mathematics*, Vol. XXIX (1899), pp. 329–381, and *ibid.*, Vol. XXXII (1901), pp. 209–239.

†† "Sulla razionaliti dei piani multipli $\{xy\sqrt[n]{F(x,y)}\}$," *Giornale di Matematiche*, Vol. XLI (1903), pp. 285–320, and also "Sulla razionaliti dei piani multipli $\{x, y\sqrt[n]{F(x,y)}\}$," *Annali di Matematica*, Series 3, Vol. II (1899), pp. 277–296.

birational transformation. For each type there exists in one plane a point determined by two straight lines, and in the other three image points. The five distinct types are defined in the following ways:

(1) a. A field of straight lines, $\Sigma y_i x_i = 0$.

b. A net of cubic curves $\phi_i = 0$ without basis points, $\Sigma y_i \phi_i = 0$.

(2) a. A pencil of lines, $y_1 x_1 + y_2 x_2 = 0$.

b. A net of curves of order higher than 3 and having a fixed point of multiplicity $n-3$ at the vertex of the pencil of lines $y_1 x_1 + y_2 x_2 = 0$. This is associated with Jonquières' configuration in (1, 1) and (1, 2) transformations. A discussion of the (1, 1) type has been presented by P. P. Boyd.*

(3) a. A net of conics through one fixed point.

b. A net of conics through the same fixed point,

$$\Sigma y_i u_i = 0, \quad \Sigma y_i v_i = 0,$$

where $u_i = 0$, $v_i = 0$ are general conics through a fixed point.

(4) a. A net of cubic curves passing through six fixed points.

b. A net of cubic curves passing through the same fixed points,

$$\Sigma y_i u_i = 0, \quad \Sigma y_i v_i = 0,$$

where $u_i = 0$, $v_i = 0$ now represent cubic curves through six basis points.

(5) a. A pencil of cubic curves through nine fixed points.

b. A net of curves of order 9 passing triply through eight of the nine fixed points,

$$y_1 \psi_1 + y_2 \phi_2 = 0, \quad \Sigma y_i \psi_i = 0.$$

If u_i is of order m_1 and v_i is of order m_2, the transformation curves are of order $m_1 + m_2 = m$. To the line $y_i = 0$ correspond curves $\phi_i(x) = 0$ of order m; to the lines $x_i = 0$ correspond curves $\psi_i(y) = 0$ of order m.

The fixed points common to all curves of the defining system $u_i = 0$, $v_i = 0$ are fundamental. To each fundamental point F corresponds a curve $f(y)$ whose order is equal to the multiplicity of the point on the curves $\phi_i(x) = 0$, and to each direction through the fundamental multiple point corresponds a point on the image fundamental curve $f(y)$. The complete image of $f(y)$ is composite. The components are the original multiple point and a residual curve.

Every branch of this residual curve has contact with some branch of K through this point. To each of these simple intersections on L is related a

* "On the Perspective Jonquières Involutions Associated with the (2, 1) Ternary Correspondence." AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), pp. 290–324.

direction of J' through the original point. Aside from these intersections J' has only points of contact in common with the residual curve and these correspond to the points of tangency of L with $f(y)$. See F. Chizzoni.*

If two curves of the defining system have contact at a given point, then one curve of the net has a double point at the point. Hence the locus of all the double points of curves of the net is the curve K.

The order of the Jacobian is $3(m-1)$. The multiplicity of any fundamental point P_i on the $\phi_i=0$ which lies on the Jacobian is $3i-1$.

The equation of L may be obtained by eliminating (x_i) from the equation for K and the equations of transformation $y_i=\phi_i(x)$.

The order of the curve L is the number of intersections aside from those at the fundamental points of K with the image of a straight line (y). This has been expressed by Zeuthen† in the formula $2(k+p-1)$ where k is the multiplicity of the transformation and p is the genus of the curves of transformation, images of the straight lines in (y).

The complete image in (x) of L is a composite curve which has for its order the product of the orders of L and the transformation ϕ. The image contains K twice and the residual curve J', and also contains fundamental curves, when L passes through their corresponding fundamental points.

Type I.

3. The defining equations for this type are

$$x_1y_1+x_2y_2+x_3y_3=0, \quad u_1y_1+u_2y_2+u_3y_3=0,$$

where $u_i=0$ are cubic curves having no fixed point in common. On solving for y_i we obtain

$$\rho y_1=u_3x_2-u_2x_3=\phi_1(x), \quad \rho y_2=u_3x_1-u_1x_3=\phi_2(x), \quad \rho y_3=u_1x_2-u_2x_1=\phi_3(x).$$

To obtain the points common to the three curves $\phi_i=0$ we write the equations in the form $\frac{u_2}{x_2}=\frac{u_3}{x_3}=\frac{u_1}{x_1}$. Of the sixteen intersections of any two curves as $\phi=0$, $\phi_3=0$ we exclude those which make $u_1=0$, $x_1=0$ as these points do not lie on $\phi_1=0$. Hence we find $16-3=13$ points common to all the curves of the net. These are simple fundamental points.

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* "Sopra le involuzioni piane," *Lincei Memoire*, Series 3, Vol. XIX (1884), p. 301.

† "Nouvelle démonstration de théorèmes sur les séries de points correspondants sur deux courbes," *Mathematische Annalen*, Vol. III (1871), p. 150.

The number of conditions required to determine a quartic curve is fourteen. Since there are thirteen fixed points on each quartic curve, any one of the three variable points uniquely fixes the other two.

The genus of the curves $\phi_i=0$ is 3 since they have no double points. The image curve is a rational quartic curve and has a triple point at (a_1, a_2, a_3). Two of these quartic curves intersect in sixteen points, each of which has three image points in (x). The complete image of this quartic curve is a curve of order 16. This curve is composed of the original line and a curve of order 15. Two of these curves intersect in two hundred and twenty-five points. But if each complete curve passes through each of the thirteen fundamental points four times, aside from these intersections there are $225-4\cdot 4\cdot 13=17$ points. But the two straight lines intersect in one image point, and each line cuts the curve of order 15 in fifteen points. The forty-eight image points are made up of seventeen points on the two curves of order 15, not at the thirteen fundamental points; one point of intersection of the two straight lines and fifteen points on each line through the curve of order 15 with which it is related.

The curve K is of order 9 with double points at the thirteen basis points. Hence it is of genus 15. The number of variable intersections of K with the image in (x) of a straight line in (y) determines the order of the image of K. The curve L is, then, of order 10. $9\cdot 4-2\cdot 13=10$. It is also of genus 15, hence it has twenty-one double points or their equivalents. The curve J' is of the order 22 and genus 15, hence has one hundred and ninety-five double points or their equivalents.

The number of intersections of K with J' aside from those at the fixed points are forty-two, or twenty-one contacts, $9\cdot 22-13\cdot 2\cdot 6$. But this is the number of double points of L to be accounted for.

It will be shown that the forty-two intersections of K with J' are twenty-one contacts, images of cusps on L. The equation of L is the condition that a line in (x) touches its associated cubic, hence it is the discriminant of a binary cubic equated to zero. These values that make the binary cubic in x_2, x_3 a perfect cube correspond to cusps on L since the form of the discriminant is $4G^2+H^3$. Any point on L that makes $H=0$ and $G=0$ is such that all three images of the point are coincident. Hence in (x) K and J' have the image point in common.

The cuspidal tangent to L corresponds to the particular quartic of the net which has a node at the corresponding point on K.

Fundamental System.

4. Each of the thirteen points of the net of quartics when substituted in the defining equations, makes the two equations identical, hence they are all fundamental points, and their images are straight lines in (y). There are no fundamental points in (y), hence no fundamental curves in (x).

The complete image of each of these lines is a curve in (x) of order 4 which has a double point at the original point. For, let $\bar{x}_1 y_1 + \bar{x}_2 y_2 + \bar{x}_3 y_3 = 0$ be the equation of the fundamental line in (y) which is the image of the fundamental point $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ which is any of the thirteen points (basis) of the net of quartics.

The image of this in (x) is obtained by making the transformation according to the equations already obtained and we have

$$\bar{x}_1(u_3 x_2 - u_2 x_3) + \bar{x}_2(u_1 x_3 - u_3 x_1) + \bar{x}_3(u_2 x_1 - u_1 x_2) = 0.$$

Differentiating this with regard to x_1 we have,

$$\bar{x}_1\left(x_2 \frac{du_3}{dx_1} - x_3 \frac{du_2}{dx_1}\right) + \bar{x}_2\left[x_3 \frac{du_1}{dx_1} - \left(u_3 + x_1 \frac{du_3}{dx_1}\right)\right] + \bar{x}_3\left(u_2 + x_1 \frac{du_2}{dx_1} - x_2 \frac{du_1}{dx_1}\right) = 0,$$

substituting in this the coordinates of the point (x_1, x_2, x_3), we have the form written as the expression

$$\bar{x}_1\left(\bar{x}_2 \frac{d\bar{u}}{dx_1} - x_3 \frac{d\bar{u}_2}{dx_1}\right) + x_2\left(\bar{x}_3 \frac{d\bar{u}_1}{dx_1} \bar{u}_3 - \bar{x}_1 \frac{du_3}{dx_1}\right) + x_3\left(\bar{u}_2 + x_1 \frac{d\bar{u}_2}{dx_1} - x_2 \frac{d\bar{u}_1}{dx_1}\right) = 0.$$

But the only terms which do not vanish are $\bar{u}_2\bar{x}_3 - \bar{u}_3\bar{x}_2 = 0$, which equation is satisfied since $(\bar{x})$ is on $u_2 x_3 - u_3 x_2 = 0$. This quartic curve has one double point and twelve simple fundamental points. The intersections of K and J' with this image curve aside from those at the fundamental points are eight $(9 \cdot 4 - 2 \cdot 2 - 2 \cdot 12 = 8)$ for K, and four for J' $(4 \cdot 22 - 2 \cdot 6 - 6 \cdot 12 = 4)$.

At the common double point K touches each branch of the quartic. This is shown as follows: When two of the image points coincide, two or more of the curves are tangent at the basis points of the net and one curve has a double point. If the Jacobian has a double point at a basis point, it is tangent to each branch of the nodal curve of the net at this point.

The Jacobian can then be formed from a net of curves, one of which has at a fundamental point $(\bar{x}_1) = (0, 0, 1)$, a double point. These curves are

$$\begin{aligned}
\psi_1 &= (x_1x_2)^2 x_3^2 + (x_1x_2)^3 x_3 + (x_1x_2)^4 = 0,\\
\psi_2 &= u_1(x_1x_2)x_3^3 + u_2(x_1x_2)x_3^2 + u_3(x_1x_2)x_3 + u_4(x_1x_2) = 0,\\
\psi_3 &= v_1(x_1x_2)x_3^3 + v_2(x_1x_2)x_3^2 + v_3(x_1x_2)x_3 + v_4(x_1x_2) = 0.
\end{aligned}$$

The Jacobian is of the form

$$\begin{vmatrix} x_1x_2^2x_3^2+- & x_2(x_1)_2x_3^2+-- & 2(x_1x_2)_2x_3+(x_1x_2)x_3+-- \\ u_1(x_1x_2)x_3^2+- & u_2(x_1x_2)x_3^2+-- & 3u(x_1x_2)x_3+u_1(x_1x_2)_3+-- \\ v_1(x_1x_2)x_3^2+- & v_2(x_1x_2)^2x_3^2+-- & 3v(x_1x_2)_2x_3+v_1(x_1x_2)_3+-- \end{vmatrix}.$$

By collecting the coefficients of x_3 the expanded form is found to be

$$3x_2u_2v-3x_2v_2u-3u_1x_1v+2u_1v_2x_1x_2+3x_1v_1u-2u_2v_1x_1x_2=0,$$

or

$$3v(x_2u_2-u_1x_1)-3u(x_2v_2-x_1v_1)+2x_1x_2(u_1v_2-u_2v_1)=0.$$

But

$$u=ax_1+bx_2,\quad \frac{du}{dx_1}=a=u_1,\quad \frac{du}{dx_2}=b=u_2,$$

and

$$v=cx_1+dx_2,\quad \frac{dv}{dx_1}=c=v_1,\quad \frac{dv}{dx_2}=d=v_2.$$

The Jacobian becomes

$$3(cx_1+dx_2)(bx_2-ax_1)-3(ax_1+bx_2)(dx_2-cx_1)+2x_1x_2(ad-bc)=0,$$

or

$$3[bcx_1x_2+bdx_2^2-acx_1^2-adx_1x_2-adx_1x_2-bdx_2^2+acx_1^2+bcx_1x_2 \\ +2x_1x_2(ad-bc)]=0,$$

or finally $$2x_1x_2(bc-ad)=0.$$

This is the coefficient of the highest power which does not vanish in the equations of the tangents to the curves of the net passing through (x_1) and is also the coefficient of highest power of x_3 in the equation of the Jacobian. It is the form of the tangent to the Jacobian also, and we have proved that the nodal ϕ has the same tangent as the Jacobian at the double point. These are $x_1=0$ and $x_2=0$.

Each direction of J' through the point corresponds to one point of intersection of L with the fundamental line. The four remaining points of intersection on J' are two contacts, the images of the two points of contact of L with the fundamental line.

The line in (y) is bitangent to L. The six simple intersections are images of the six directions on J' at the fundamental point.

Type II.

5. The defining equations for Type II are

$$x_1y_1+x_2y_2=0, \tag{1}$$

$$u_1y_1+u_2y_2+u_3y_3=0, \tag{2}$$

where $u_i=0$ represent curves of order $n>3$ having an $n-3$-fold point in common. Let this point be $P(x)=(0,0,1)$. Then

$$\rho y_1=u_3x_2=\phi_1(x), \quad \rho y_2=-u_3x_1=\phi_2(x), \quad \rho y_3=u_2x_1-u_1x_2=\phi_3(x).$$

These curves $\phi_i=0$ have an $(n-2)$-fold point at $P(x)$. The equation of the image of a general line in (x) has the form

$$\begin{aligned}(\text{a}) \quad & y_1\{(a_1y_2-a_2y_1)^3[f_1(a_3-y_1y_{2_{n-3}})]+(a_1y_2-a_2y_1)^2[f_1(a_3-y_1y_{2_{n-2}})]\\ &+(a_1y_2-a_2y_1)[f_1(a_3-y_1y_{2_{n-1}})]+f_1(a_3-y_1y_{2_n})\}\\ &+y_2\{(--)\}+y_3\{(--)\}=0.\end{aligned}$$

Let

$$\begin{aligned} f_i(a_3,\, -y_1y_{2_{n-3}})&=v_i, & f_i(a_3,\, -y_1y_{2_{n-1}})&=s_i,\\ f_i(a_3,\, -y_1y_{2_{n-2}})&=w_i, & f_i(a_3,\, -y_1y_{2_n})&=t_i,\end{aligned}$$

the equation of the curve is then

$$\begin{aligned}(\text{b}) \quad & (a_1y_2-a_2y_1)^3(y_1v_1+y_2v_2+y_3v_3)+(a_1y_2-a_2y_1)^2(y_1w_1+y_2w_2+y_3w_3)\\ &+(a_1y_2-a_2y_1)(y_1s_1+y_2s_2+y_3s_3)+(y_1t_1+y_2t_2+y_3t_3)=0.\end{aligned}$$

This curve is of order $n-3+3+1=n+1$ having at $P(y)$ an n-fold point.

Aside from the intersections at $P(y)$ two curves of order $n+1$ will have $(n+1)^2-(n)^2=2n+1$ points in common, of which one is variable.

The image of a line through $P(x)$ is found by dividing the above equation (a) by a_3^{n-3} and then putting $a_3=0$. All terms containing a_3 to a power greater than $n-3$ will vanish. The remaining terms have the form

$$(a_1y_2-a_2y_1)^3[y_1f(y_1y_{2_{n-3}})+y_2f_2(y_1y_{2_{n-3}})+y_3f_3(y_1y_{2_{n-3}})]=0,$$

which consists of a line $a_1y_2-a_2y_1=0$ through $P(y)$ taken three times, and a curve $\Sigma y_if_i(y_1y_{2_{n-3}})=0$ of order $n-2$, having at $P(y)$ an $n-3$-fold point, and is independent of the coefficients a_i. The line is the image of the given line, and the curve is the image of the point $P(x)$.

The Fundamental System.

6. The $6(n-1)$ basis points are all fundamental; their images in (y) are straight lines through $P(y)$.

The complete image in (x) of each of these fundamental lines in (y) is a curve of order $n+1$ having an $(n-2)$-fold point at $P(x)$. But, since the line in (y) passes through $P(y)$, the image of $P(y)$ must be deducted. It is a curve C_1 passing through $P(x)$, and the one point of the $6(n-1)$ basis points of which the line in (y) is an image. The proper image in (x) of the line in (y) is, then, of order n having an $n-3$-fold point at $P(x)$ and passing once through each of the $6(n-1)$ points.

The point $P(x)$ of order $n-2$ on the transformation $\phi_i=0$ has a curve of order $n-2$ as its image in (y), with a point of multiplicity $n-3$ at $P(y)$. The complete image of this curve in (x) is of order $(n+1)(n-2)-n(n-3)=2(n-1)$, having at $P(x)$ a multiplicity $2n-4$ and passing once through each of the $6(n-1)$ points.

The curve K is of order $3n$, having at $P(x)$ a point of multiplicity $n-2+n-2+n-3=2n-7$. But u_3 is a factor of each term, so that the curve K is of order $2n$, having a $2n-4$-fold point. It also passes simply through the $6(n-1)$ simple basis points. The genus of K is $6n-9$.

The order of L is $2n(n+1)-(n-2)(2n-4)-6n-6=4n-2$, and its genus is $6n-9$. The curve has at $P(y)$ a point of multiplicity $4n-6$ and a double point at each of the $6(n-1)$ basis points. The curves K, J' have $6(n-1)$ contacts and L has $6(n-1)$ cusps.

Let P_i denote a simple fundamental point in (x); p_i denote the image in (y) of P_i, and let r_i denote the residual image curve in (x) of p_i.

Thus J' has a double point at P_i and one contact not at the fundamental point corresponding to the tangency of L with p_i.

We denote the point in (x) by Q, its image curve in (y) by q and the residual curve in (x) by s.

K passes through Q $2n-4$ times and is tangent to each of the $n-2$ branches of ϕ_i. There are $2n-4$ points of tangency of L with q, leaving $4n-6$ simple intersections. These simple intersections correspond to the directions of J' through Q.

The intersections of K with s, not at the fundamental points are shown to be $4n-6$. These intersections correspond to the simple intersections of L with q. Aside from the fundamental points, J' has $4n-8$ intersections or $2n-4$ contacts with s which correspond to the points of contact of L with q.

Type III.

7. The defining equations for Type III are

$$u_1y_1+u_2y_2+u_3y_3=0, \qquad (1)$$

$$v_1y_1+v_2y_2+v_3y_3=0, \qquad (2)$$

where $u_i=0$, $v_i=0$ are general conics through $P=(0,0,1)$.

The Fundamental System.

8. The image of a point $P(x)$ is a curve $p(y)$ of order 2 in (y). The complete image in (x) of $p(y)$ is a curve of order 8 having a five-fold point at $P(x)$ and double points at each of the nine simple basis points.

5

The image of each of the nine simple basis points in (y) is a line whose complete image in (x) is a curve of order 4 belonging to the net having a double point at $P(x)$ and also at the given basis point.

K is of order 9, genus 9. There are no fundamental points in the (y) plane. The complete image of K consists of the conic $p(y)$ taken five times; images of the double point $P(x)$ through which K passes five times; nine straight lines each counted twice, images of the nine simple basis points, through which K passes twice, and also the curve of branch points L.

L is of order 8, genus 9; it has twelve cusps. J' has a six-fold point at $P(x)$; J' and K have twelve points of contact.

We shall denote one of the nine basis points of the net by Q_i, the straight line in (y) which is its image by q_i and the residual image curve in (x) by s. We shall denote the image of the fundamental point $P(x)$ of multiplicity 2 on ϕ_i by p and its residual curve in (x) by r.

As in Type I, L is tangent to q as many times as K passes through Q. Since K has a double point at each of the nine simple basis points, q is tangent to L twice. L and q have $1\cdot 8-2\cdot 2=4$ simple intersections, each of which corresponds to a direction of J' through Q.

K has four simple intersections with s not at fundamental points, the images of the four simple intersections of L with q.

J' has four simple intersections or two contacts with s corresponding to the two contacts of L with q.

Corresponding to the five contacts of K with r at $P(x)$ are five contacts of L with p, which leaves six simple intersections of p with L. These correspond to the directions of J' through $P(x)$.

K has six simple intersections with r besides those at the fundamental points. These are images of the simple intersections of L with p.

J' has ten simple intersections or five contacts with r not at fundamental points. These correspond to the five contacts of L with p.

In the general case of two nets of conics through a common point we have the configuration of quartic curves having a double point and nine simple points. As a subcase is included a system of conics having a simple point in common and no basis points. This system is a linear combination of conics forming a net.

Type IV.

9. The defining equations for Type IV are

$$u_1y_1+u_2y_2+u_3y_3=0, \qquad (1)$$

$$v_1y_1+v_2y_2+v_3y_3=0, \qquad (2)$$

where $u_i=0$, $v_i=0$ are cubic curves passing through six fixed points. We shall denote these points by P_i.

The equations of transformation are

$$\rho y_1=u_2v_3-u_3v_2=\phi_1(x), \text{ etc.}$$

The curves ϕ_i have a double point at each of the six P_i. They also intersect in $6\cdot 6-6\cdot 2\cdot 2-3=9$ simple points of the system. We shall denote these basis points by Q_i. Each ϕ_i is of genus 4. The image in (y) of a line in (x) is a sextic curve.

The Fundamental System.

10. Each P_i of the net as well as each Q_i when substituted in the equations (1) and (2) make them identical. Hence they are fundamental points.

The image of each P_i in (y) is a conic which we shall denote by p_i. The residual image of $p_i(y)$ in (x), *i. e.*, r_i is a curve of order 12 having at the given P_i a five-fold point. The residual curve r_i has a double point at Q_i.

The image of each Q_i is a straight line q_i in (y) and a residual curve s_i in (x) of order 6 belonging to the net. At the given Q_i the curve r_i has a double point and a simple point at each of the remaining eight Q_i. The curve r_i has a double point at each P_i.

K is of order 15, genus 22; L is of order 12, and has thirty-three cusps. Aside from L the complete image of K is composed of nine straight lines taken twice, images of the nine basis points through which K passes twice and six conics taken five times, images of the six P_i through each of which K passes five times.

The complete image of L in (x) is a curve of order 72 of which K is a double component. The residual image J' is of order 42. The complete multiplicity of each P_i on K^2J' is 24 from which a five-fold point taken twice must be deducted for K^2. Hence P_i is of multiplicity 14 on J'. The total number of intersections at each Q_i is twelve, of which four are on K^2. J' has at each Q_i an eight-fold point. The genus of J' is 22. K and J' intersect in $42\cdot 15-6\cdot 14\cdot 5-9\cdot 2\cdot 8=66$ points not at fundamental points which are thirty-three contacts corresponding to the thirty-three cusps on L.

Each q_i meets L in twelve points. As Type I, L is tangent to q_i as many times as K is multiple at Q_i. Hence there are two contacts of L with q_i and eight simple intersections each of which corresponds to a particular direction of J' through Q_i.

K intersects r_i in eight points not at fundamental points which correspond to the eight simple points of intersection of L with q_i.

J' has four simple intersections or two contacts with r_i which are images of the two contacts of L with q_i.

In the same way L meets p_i in twenty-four points of which five are contacts and fourteen are simple intersections. The points of tangency correspond to the contacts of K with s_i at P_i. Each of the simple intersections corresponds to a particular direction of J' through P_i.

There are fourteen intersections of K with s_i not at fundamental points, images of the fourteen intersections of L with p_i.

There are ten intersections of J' with s_i, not at fundamental points, which are five contacts corresponding to the five contacts of L with p_i.

Type V.

11. The defining equations for Type V are

$$u_1y_1+u_2y_2+u_3y_3=0, \tag{1}$$

$$v_1y_1+v_2y_2+v_3y_3=0, \tag{2}$$

where $u_i=0$ represent cubic curves passing through nine fixed points, and $v_i=0$ represent curves of order 9, having a triple point at eight of the nine fixed points of $u_i=0$. The equations of transformation are

$$\rho y_1=u_2v_3=\phi_1(x),\quad \rho y_2=-u_1v_3=\phi_2(x),\quad \rho y_3=u_2v_1-u_1v_2=\phi_3(x).$$

There are eight points P_i of multiplicity four on each ϕ_i, and thirteen simple basis points Q_i. The fundamental curve $v_3=0$ has a three-fold point at each P_i and passes simply through twelve of the thirteen Q_i. $9\cdot 12-8\cdot 3\cdot 4=12$. We shall denote by Q_m the basis point through which $v_3=0$ does not pass.

The Fundamental System.

12. The image in (y) of each fundamental point Q_i is a straight line q_i passing through $P(y)$. The residual image in (x) is a composite curve of order 12 having eight points of multiplicity, four at $P(x)$, passing simply through eleven of the Q_i, and doubly through the given Q_i. One component

of this curve is $v_3=0$. The remaining image r_i is a curve of order 3 having a simple point at each P_i and also at the given Q_i and Q_m.

The image in (y) of Q_m is a straight line q_m which does not pass through $P(y)$. The residual image r_m in (x) is a curve of order 12 having a four-fold point at each P_i, a simple point at each Q_i, and a double point at Q_m.

The image of each P_i is a curve p_i of order 4 having a three-fold point at $P(y)$. The residual image in (x) is a composite curve of order 48, having a sixteen-fold point at seven P_i and a seventeen-fold point at the given P_i. The multiplicity at Q_i and Q_m is 4. One component is the image of $P(y)$ taken as many times as p_i passes through $P(y)$. The remaining component which is the curve s_i of order 21, having a multiplicity of 7 at seven of the P_i and of 8 at the given P_i. At each Q_i there is a simple point, but at Q_m there is a four-fold point.

The Jacobian is of order 33. This is a composite curve, one component being $v_8=0$. Hence the curve of coincidences K is of order 24, having eight-fold points at P_i, simple points at Q_i and a double point at Q_m. The genus of K is 28. The order of L is 18. Corresponding to each intersection of K with $v_3=0$, not at the fundamental points, is a branch of L through $P(y)$. $24\cdot 9-8\cdot 8\cdot 3-12\cdot 2=12$. Hence L has a twelve-fold point at $P(y)$. The genus of L is 28, and there are forty-two double points, or their equivalents, to be accounted for. The curve J' is of order 60.

The total multiplicity at P_i is 72, at Q_i the total multiplicity is 18, and at Q_m it is 18. Deducting the multiplicities at these points on K^2 and $v_3^{12}=0$ there remains a twenty-fold point at P_i, a four-fold point at Q_i, and a fourteen-fold at Q_m on J'. K and J' have, aside from the fundamental points, forty-two contacts which correspond to the forty-two cusps on L.

As in Type I, L is cut by q_m in eighteen points, two of which are contacts corresponding to the tangency of K with r_m at Q_m. The fourteen simple intersections of K with r_m, not at fundamental points, are images of the fourteen simple intersections of q_m with L.

The four points of intersection of J' with r_m, not at fundamental points, are two points of contact, images of the two contacts of L with r_m.

L has six points in common with q_i, not at $P(y)$, of which one is a point of contact and four are simple intersections. These correspond to the tangency of K with r_i at Q_i and to the directions of J' through Q_i.

The intersections of K with r_i, not at fundamental points are four, the images of the four intersections of L with q_i.

The intersections of J' with r_i, not at fundamental points, are two or one point of contact which is the image of the contact of L with q_i.

L has thirty-six intersections with p_i, not at $P(y)$. Eight of these are contacts and twenty are simple intersections. These correspond to the tangency of K with s_i at P_i and the multiplicity of J' at P_i.

There are twenty simple intersections of K with s_i, not at fundamental points, which are images of the simple intersections of L with p_i.

There are sixteen intersections of J' with s_i, not at fundamental points, which are eight contacts, images of eight contacts of L with p_i.

There are eight four-fold points and thirteen simple basis points in the general case of a system composed of a net of cubic curves, and a net of curves of order 9. A subcase of this is a linear system of curves of order 9 having eight triple points and six simple basis points. Hence the latter type which is used here is a subcase of the general system composed of the net of cubics and the net of curves of order 9.

Completeness of Enumeration.

13. The three images of a point in (y) may be defined as the variable intersections of systems of curves in (x) in many ways. They are, however, reducible to one of the five types described above. For example, the defining equations may be

$$u_1y_1+u_2y_2+u_3y_3=0, \tag{1}$$

$$v_1y_2+v_2y_2+v_3y_3=0, \tag{2}$$

where $u_i=0$ represent cubic curves having a double point at $P(x)$ and a simple point at $Q(x)$, and where $v_i=0$ represent conics passing through P and Q. $u_i=0$ have the form $x_3(x_1x_2)_2+x_1(x_1x_2)=0$, and the curves $v_i=0$ have the form $x_3(x_1x_2)+x_1(x_1x_2)=0$. In $\Sigma u_i=0$ there are ∞^5 degrees of freedom, and in $\Sigma v_i=0$ there are ∞^3 degrees of freedom.

If the $u_i=0$ have the added restriction that they pass simply through a third point $R=(1,0,0)$ through which $v_i=0$ do not pass, the net has ∞^2 degrees of freedom. We shall denote the net of these cubic curves through one double and two simple points by $y_1\psi_1+y_2\psi_2+y_3\psi_3=0$. By a quadratic inversion the ψ_i becomes curves of order 6 which contain the image of the double point P and the two simple points Q and R. The residual curve is of order 2 which passes once through P but not through R or Q.

The conic of the net $v_i=0$ through P and Q become composite curves of order 4 whose residual image is a conic passing through P and Q.

The curves of transformation $\rho y_i = \phi_i$ of the system $\Sigma\psi_i y_i = 0$, $\Sigma v_i y_i = 0$ obtained by finding the values of y_i are of order 5 having at P, and Q points of multiplicity 3 and 2 and of multiplicity 1 at the nine fixed basis points T_i. By a quadratic inversion the ϕ_i are of order 10. These are composite curves whose residual images are quartic curves, having one double point and nine simple points. The genus of the quintic is 2 as is also that of the quartic.

The curve of coincidences is of order 11 having at P, Q, and T_i multiplicities of 7, 4 and 2. By the inversion K is a composite curve of order 22 whose residual image is a curve of order 9 with a point of multiplicity 2 and 5 at P and T_i. The genus of K is 9.

Hence the ψ_i reduce to conics through one fixed point, v_i become conics through this same fixed point and a point M through which the first conics do not pass. The transformation ϕ_i reduce to quartics having a double point and nine simple points. K reduces to a curve of order 9 and of genus 9 with one five-fold point and nine double points. But this is exactly the configuration for Type III. Thus the system of a net of curves of order 3 having a fixed double point and a fixed simple point through which pass a net of conics reduces by a birational transformation to the system of Type III.

The same method of reduction can be employed in every case. The procedure is exactly that used by Bertini in the paper cited.

Cyclic Involutions.

14. In order that the discussion of the (1, 3) correspondences may be complete it is necessary to consider those cases in which the images of a point (y) are rationally separable such that to a point $P(y)$ correspond three points $P_1(x)$, $P_2(x)$, $P_3(x)$ having the following property. If P_1 describe a locus, P_2 describes another locus and P_3 another locus, all in (1, 1) correspondence. If P_1 and P_2 describe the same locus, P_3 describes the same locus. Hence a birational transformation exists by means of which the three points P_i are permuted among themselves. By this transformation $P_1 \backsim P_2$, $P_2 \backsim P_3$, $P_3 \backsim P_1$. Since the three image points form a group the transformation is cyclic and of order or period 3. Such an involution is called cyclic.

If two of the three image points coincide all three are identical. For the transformation which sends P_1 into P_2 also sends P_2 into P_3 and P_3 into P_1. Hence if $P_1 = P_2$ then $P_1 = P_2 = P_3$. Hence K and J' are identical. There is a birational relation between the curves K and L.

Bottari* has discussed the possible forms of all cyclic types by means of projection from configurations in space of three or more dimensions.

For a cyclic transformation of order 3 there are four types among which are included all the Cremona transformations of order 3.

(1) The Jonquières transformation of three points on a fixed line.

(2) The Jonquières transformation of three points on a pencil of lines p_i, the lines themselves being permuted by a linear transformation of order 3. In particular P_1 on p_1 is transformed into P_2 on p_2, P_2 on p_2 is transformed into P_3 on p_3 which in turn becomes the original point. In the same way P_2 and P_3 on p_1 have uniquely associated points on p_2 and p_3 such that the three points on each p_i form a distinct group.

(3) The three permuted points are the three intersections of two cubic curves through six basis points which form two triads of points in three-fold perspective. The transformation is accomplished by means of a quartic curve through six points. One triad consisting of double points, the other of simple points. Kantor† calls the type Δ_3.

(4) A particular case of Type V, in which the curves of order 9, and a pencil of cubic curves are invariant under a Cremona transformation of period 3 and order 13. This type is called N_3 by Kantor (*loc. cit.*, p. 288).

Cyclic Type I.

15. Given a curve C_n with a fixed point of multiplicity $n-3$. The three points in which a line through the fixed point cuts the curve are images of the point of intersection of two lines in (y). The points of each triad are on an invariant line.

The equations of the transformation have the form

$$\rho x_1' = \frac{ax_1+b}{cx_1+d}, \quad \rho x_2' = x_2, \quad \rho x_3' = x_3.$$

It is of period 3 when $ad+bc+a^2+d^2=0$. Equating T^2 and T^{-1} we have

$$x^2(acd+cd^2)+x(d^3-abc)-bcd-bd^2$$
$$=x^2(-a^2c-bc^2)+x(a^3-bcd)+a^2b+abd+abd.$$

The condition for this equality to exist is found by putting

$$b=\frac{-(a^2+d^2+ad)}{c}.$$

* "Sulla razionalita dei piani multipli $\{x, y, \sqrt[n]{F(x,y)}\}$." *Annali di Matematica.* Series 3, Vol. II (1899), p. 278.

† "Premiers Fondements pour une Théorie des Transformations Periodiques Univoques," *Memoire, della R. Accademia.* Naples. Series 2, Vol. III, e Vol. IV, 2 (1891), pp. 1-335. See pp. 260-262.

In order to obtain the relations of the six points in (x) under the transformation, consider three planes x, x', x''. Under T the six (x') images of the six fundamental points (x) are obtained. These under T^2 define six points (x'') and these in turn go into the original (x) points. The (x') and (x'') planes are superposed on the (x) plane to obtain the required relations.

We shall consider the two triads of points in (x) and (x') as

$$(A_2B_2C_2\bar{A}_1\bar{B}_1\bar{C}_1) \text{ and } (A'_2B'_2C'_2\bar{A}'_1\bar{B}'_1\bar{C}'_1).$$

The points A_2, B_2, C_2 each have conics in (x') and the points $\bar{A}_1$, $\bar{B}_1$, $\bar{C}_1$ have straight lines in (x') for their images. Also the point

$$\begin{array}{llll} A'_2 \text{ goes into } C_2(ABC\bar{B}\bar{C}), & \bar{A}'_1 \text{ goes into } C_1(BC), \\ B'_2 \quad \text{"} \quad C_2(ABC\bar{A}\bar{C}), & \bar{B}'_1 \quad \text{"} \quad C_1(AC), \\ C'_2 \quad \text{"} \quad C_2(ABC\bar{A}\bar{B}), & \bar{C}'_1 \quad \text{"} \quad C_1(AB). \end{array}$$

Since any straight line in (x') goes into a quartic through $(A_2B_2C_2\bar{A}_1\bar{B}_1\bar{C}_1)$, the image of the line $A'\bar{A}'$ is a quartic curve containing the image of A' and $\bar{A}'$. These are the straight line BC and the conic through $ABC\bar{B}\bar{C}$. The residual image is, then, the line $A\bar{A}$. In the same way the images of other lines may be obtained.

Since $CB \equiv x_1=0$, $AC \equiv x_2=0$, $AB \equiv x_3=0$, it follows that

$$\begin{aligned} x_1 &= k(bcx_1^2-a^2x_2x_3)(abx_3^2-c^2x_1x_2), \\ x_2 &= l(acx_2^2-b^2x_1x_3)(bcx_1^2-a^2x_2x_3), \\ x_3 &= m(acx_2^2-b^2x_1x_3)(abx_3^2-c^2x_1x_2). \end{aligned}$$

The points P, Q, and R are invariant under the transformation, hence the substitution of the coordinates of one of these points in the above equations gives the values of k, l, and m. These are ω^2ab, ωbc and ac, respectively.

The equations of transformation T^2 are

$$\begin{aligned} x'_1 &= ab(bcx_1^2-a^2x_2x_3)(abx_3^2-c^2x_1x_2), \\ x'_2 &= bc(acx_2^2-b^2x_1x_3)(bcx_1^2-a^2x_2x_3), \\ x'_3 &= ac(acx_2^2-b^2x_1x_3)(abx_3^2-c^2x_1x_2). \end{aligned}$$

Similarly for T we obtain

$$x'_1 = ax_1x_2(abx_3^2-c^2x_1x_2), \quad x'_2 = bx_2x_3(bcx_1^2-a^2x_2x_3), \quad x'_3 = cx_1x_3(acx_2^2-b^2x_1x_3).$$

The nine points A, B, C, $\bar{A}$, $\bar{B}$, $\bar{C}$, P, Q, R lie on a cubic curve which is invariant, but since two triads of lines pass through these points, they are the basis points of a pencil of cubic curves. The equations have the form

$$A\bar{A}\cdot B\bar{C}\cdot \bar{B}C+\lambda C\bar{C}\cdot \bar{A}B\cdot A\bar{B}. \tag{1}$$

If any curve of the pencil is invariant under T, we must have

$$C\overline{C}\cdot\overline{A}B\cdot\overline{B}A+\overline{\lambda}B\overline{B}\cdot A\overline{C}\cdot C\overline{A}=0, \tag{2}$$

such that (1) and (2) define the same curve. Equation (1) has the form

$$(bx_3-cx_2)(\omega bx_1-ax_2)(\omega cx_1-ax_3)+\lambda(\omega^2bx_1-ax_2)(cx_1-ax_3)(\omega cx_2-bx_3)=0, \tag{3}$$

which by T goes into

$$(\omega^2bx_1-ax_2)(ax_3-cx_1)(bx_3-\omega cx_2)+\overline{\lambda}(ax_3-\omega^2cx_1)(\omega bx_3-cx_2)(bx_1-ax_2)=0. \tag{4}$$

By comparing the coefficients of like terms in equations (3) and (4) we obtain the relation $\overline{\lambda}=\dfrac{\omega}{\omega+\lambda}$. By the substitution of $\dfrac{\omega}{\omega+\lambda}$ for $\overline{\lambda}$ in equation (4) we find $\lambda=\omega$ and $\lambda=\omega^2$. The two curves are

$$bc^2x_1^2x_2+\omega ca^2x_2^2x_3+\omega^2ab^2x_1x_3^2=0, \tag{5}$$

$$b^2cx_1^2x_3+\omega^2ac^2x_1x_2^2+\omega a^2bx_2x_3^2=0. \tag{6}$$

The existence of the two cubic curves satisfied by different values of λ has been mentioned by H. S. White.*

It is necessary to determine whether either of these cubic curves contains points invariant under the transformation. The point of contact of a line through $A=(0,0,1)$ or $x_1=mx_2$ with (5) gives $x_1=\omega a\sqrt[3]{4}$, $x_2=b\sqrt[3]{4^2}$, $x_3=-2\omega c$. Under T these values become $x_1=(\sqrt[3]{4^2})\omega a$, $x_2=-2\omega b$, $x_3=\sqrt[3]{4}c$. Hence this point is not invariant. The point of contact of $x_1=mx_2$ with (6) gives $x_1=a\sqrt[3]{4^2}$, $x_2=b\omega^2\sqrt[3]{4}$, $x_3=-2c$. Under T this point is invariant, hence the curve (6) is the equation of $K=J'$.

All the cubic curves through $A, B, C, \overline{A}, \overline{B}, \overline{C}$ which contain the three permuted points which are images of a point (y) form a net of equianharmonic curves; they are tangent to one of the sides $x_1=0$, $x_2=0$, $x_3=0$ and do not pass through P, Q and R.

Since a line in (y) becomes a cubic curve in (x) any one of these equianharmonic curves may be taken for $\phi_i=0$ in the equations of transformation which written in the form of defining equations are $\rho y_i=\phi_i$.

The equation of L, the image of K, is also a cubic curve. To a line in (x) corresponds a cubic in (y) which has for its complete image in (x) a composite curve made up of the original line and two quartic curves. The line when operated upon by the transformation T becomes one of these quartic curves, and when operated upon by T^2 becomes the second quartic which accounts for the curves of order 9 as $C_1+C_4+C_4$.

* "Plane Cubics and Irrational Covariant Cubics," *Transactions American Mathematical Society*, Vol. I (1900), pp. 170–178.

Non-perspective Linear Transformations.

20. It was shown in Art. 17 that quadratic transformations of period 3 can be reduced to the Jonquières type of cyclic involution.

We now consider conics through one fixed point, invariant under the non-perspective cyclic linear transformation of period 3.

If $\rho x_1'=\omega x_1$, $\rho x_2'=\omega^2 x_2$, $\rho x_3'=x_3\,(\omega^3=1)$, the expression $x_1x_2+kx_3^2$ remains invariant, $x_1x_3+lx_2^2$ is multiplied by ω^2, and $x_2x_3+mx_1^2$ is multiplied by ω by the transformation. Hence the systems of conics

$$x_1x_2y_1+x_3^2y_2=0,\quad x_2^2y_3+x_1x_3(y_1+y_2+y_3)=0$$

defines a triad of variable intersections which is invariant under the transformation.

For K we find $x_1x_2^2x_3^2(x_1x_2-x_3^2)\,(x_1x_3+x_2^2)=0$, which is satisfied only by fundamental curves. This is a particular case of Cyclic Type III.

Consider the ∞^3 system of cubic curves

$$\alpha_1x_1^3+\alpha_2x_2^3+\alpha_3x_3^3+6\alpha_4x_1x_2x_3=0.$$

Every curve goes into itself by the transformation. Any two curves of the system intersect in nine points, making three triads. Now choose any two points (x), $(\bar{x})$ as basis points of the system of cubic curves. Associated with (x) are two distinct points forming a triad, and associated with $(\bar{x})$ are two points so that the six points form two triads of permuted points. There remains one triad of variable points which are rationally distinct and are images of a point in (y).

Geometric Interpretation of Cyclic Type III.

21. The relation between any point of the (y) plane with its three corresponding images in the (x) plane has been developed for Type III by Bottari.*

Cyclic Type IV.

22. Type IV is defined by a system of curves of order 9 which are invariant under the transformation and on which lie the three variable points which are permuted among themselves.

The system has ∞^6 degrees of freedom as shown in the form

$$\lambda_1\psi_1+C_6(\lambda_2\phi_1+\lambda_3\phi_2)+\lambda_4\phi_1^3+\lambda_5\phi_1^2\phi_2+\lambda_6\phi_1\phi_2^2+\lambda_7\phi_2^3=0. \qquad (1)$$

Since ψ, $\phi_i\phi_k$ are invariant under T, but C_6 is not, we may reject the terms involving C_6,

$$\lambda_1\psi+\lambda_2\phi_1^3+\lambda_3\phi_1^2\phi_2+\lambda_4\phi_1\phi_2^2+\lambda_5\phi_2^3=0.$$

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* *Loc. cit.*, p. 282.

Two of the four constants in equation (2) may be chosen arbitrarily. We shall let one constant determine a point P on the ϕ_2 which also uniquely determines the two remaining associated points of the triad. Substituting the coordinates of this point in (2) there will be a fixed value $\lambda_1\psi_p$ and $\lambda_4\phi_p^3$ since P does not lie on ψ or ϕ_1. The resulting equation is then, $\lambda_1 k+\lambda_4 l=0$ or $\lambda_4=-\lambda_1\dfrac{k}{l}$. A second constant may be used to determine a point Q on ϕ_1 with which are also associated the two remaining points of the triad.

$$\lambda_1\psi+\phi_1^2\phi_2+\phi_1\phi_2^2-\lambda_1\frac{k_1}{l}\phi_1^3-\lambda_1\frac{k'}{l'}\phi_2^3=0,$$

or

$$\lambda_1(\psi+\phi_1^3+\phi_2^3)+\lambda_2\phi_1^2\phi_2+\lambda_3\phi_1\phi_2^2=0.$$

The curves $f_i(x)=0$ intersect into two variable triads.

The curves $f_i(x)$ intersect in six fixed points which are fundamental points, $9\cdot 9-8\cdot 3\cdot 3-3=6$ composed of two triads of points P_i and Q_i.

Through the point $(1, 0, 0)$ we have a pencil of lines of the form $\phi_1\phi_2(\phi_1 y_2+\phi_2 y_3)=0$. To the lines of the pencil correspond cubic curves in (x). The transformation curves $\rho y_i=f_i(x)$ with eight triple points and two triads of simple points have fundamental elements in (y) corresponding to their basis points. To P_i correspond lines p_i in (y) through $(1, 0, 0)$. The residual image of these lines is a curve of order 9 composed of a sextic, the image of the point, and a cubic, the image of the line. In the same way the residual image of Q_i is a cubic curve $\bar{Q}_i(x)$.

The triple points R_i on the $f_i(x)$ have for images rational curves of order 3 in (y) with a double point at $(1, 0, 0)$. The complete image in (x) is a composite curve r_i of order 27 containing the image of the fundamental point $(1, 0, 0)$ twice. The residual component $\bar{R}_i$ is of order 15, having eight points of multiplicity 5 at R_i.

The Jacobian of the system is a sextic having eight double points. It is of genus 2.

The curve L is also a sextic of genus 2. There are four branches of the curve through $(1, 0, 0)$ consisting of two sets of tangents which count for eight intersections. The common tangent for two branches can not cut the curve again, but an arbitrary line through $(1, 0, 0)$ cuts L in two other points corresponding to the two points of intersection of a cubic of the pencil with K. Two cubics do not meet K except at fundamental points.

L cuts p_i in two points not at $(1, 0, 0)$ and $\bar{P}_i$ has two intersections with K not at fundamental points. $6\cdot 3-8\cdot 1\cdot 2=2$. If p_i is tangent to L at

$(1, 0, 0)$, $\overline{P}_i$ has no simple intersection with K but one tangency. This is also true for q_i in (y) and $\overline{Q}_i$ in (x). L cuts r_i in ten points or five contacts. $6 \cdot 3 - 4 \cdot 2 = 10$. K is tangent to each branch of R at R_i which correspond to the tangency of L with r_i at the five points, and there are no simple intersections. $6 \cdot 15 - 8 \cdot 5 \cdot 2 - 2 \cdot 5 = 0$.

The Cyclic Transformation N_8.

23. The cyclic transformation of period 3 involving eight fundamental points is of order 13. By means of the transformation the image of a line is a curve of order 13 on which the eight fixed points are distributed as follows: one triad of four-fold points A, B, C; one triad of five-fold points $\overline{A}, \overline{B}, \overline{C}$; one triple point G, and one six-fold point $\overline{G}$. In order to obtain the relation of the eight P_i under the transformation we shall consider the two planes (x) and (x') on which are the points $A_4B_4C_4\overline{A}_5\overline{B}_5\overline{C}_5G_8\overline{G}_6 : A'_4B'_4C'_4\overline{A}'_5\overline{B}'_5\overline{C}'_5G'_8\overline{G}'_6$, respectively.

The image of A, B, and C are three rational quartic curves in (x') having three double points and five simple points, the image of $\overline{A}$, $\overline{B}$, and $\overline{C}$ are three rational curves of order 5 having six double points and two simple points; the image of G is a cubic curve with one double point and six simple points, and of $\overline{G}$ the image is a sextic curve having one triple point and seven double points.

The multiplicity of A, B, C on the Jacobian is 11, of $\overline{A}, \overline{B}, \overline{C}$ is 14, of G is 8 and of $\overline{G}$ is 17.

By the transformation the line $A'C'$ goes into the curve

$$C_{13}(A_4B_4C_4\overline{A}_5\overline{B}_5\overline{C}_5G_3\overline{G}_6).$$

The image of a point A' is the curve $C_4(A_1B_1C_1\overline{A}_1\overline{B}_2\overline{C}_2G_1\overline{G}_2)$ and of the point C' is the curve $C_4(A_1B_1C_1\overline{A}_2\overline{B}_2\overline{C}_1G_1\overline{G}_2)$. Hence the residual image of the line $A'C'$ is a quintic curve $C_5(A_2B_2C_2\overline{A}_2\overline{B}_1\overline{C}_2G_1\overline{G}_2)$.

In the same way the residual image of $A'B'$ is the curve

$$C_5(A_2B_2C_2\overline{A}_2\overline{B}_2\overline{C}_1G_1\overline{G}_2).$$

Consider the points (x') superposed upon the (x) points such that $A'B'C'\overline{A}'\overline{B}'\overline{C}'G'\overline{G}' \equiv \overline{B}\overline{C}\overline{A}BCA\overline{G}G$, respectively.

By T the point A' goes into

$$C_4(A_1B_1C_1\overline{A}_1\overline{B}_2\overline{C}_2G_1\overline{G}_2) \equiv C_4(\overline{C}'_1\overline{A}'_1\overline{B}'_1C'_1A'_2B'_2\overline{G}'_1G'_2).$$

By T^2 this curve becomes a composite curve of order $4 \cdot 13$ or

$$C_{52}(A_{16}B_{16}C_{16}\overline{A}_{20}\overline{B}_{20}\overline{C}_{20}G_{12}\overline{G}_{24}),$$

from which the components which are images of the points $\overline{C}'\overline{A}'\overline{B}'C'A'B'\overline{G}'G'$ are deducted as $C_{47}(A_{14}B_{14}C_{14}\overline{A}_{18}\overline{B}_{18}\overline{C}_{18}G_{10}\overline{G}_{28})$.

Hence the residual component is a curve

$$C_5(A_2B_2C_2\overline{A}_2\overline{B}_2\overline{C}_1G_3\overline{G}_1)\equiv C_5(\overline{C}_2'\overline{A}_2'\overline{B}_2'C_3'A_2'B_1'\overline{G}_2'G_1').$$

By T again the quintic becomes a composite curve $C_{65}(A_{20}B_{20}C_{20}\overline{A}_{25}\overline{B}_{25}\overline{C}_{25}G_{15}\overline{G}_{30})$, from which the image of the fundamental points of the quintic are deducted as before.

Hence this curve passes through the point $A'=\overline{B}$ once more than the parameters provide for, so that this point which is the residual image of T^3 is the original point. The transformation is, then, of period 3.

In the same way it can be shown that the point $\overline{G}'\equiv G$ by T goes into $C_3(A_1B_1C_1\overline{A}_1\overline{B}_1\overline{C}_1G_0\overline{G}_2)$. By T^2 the residual image is $C_6(A_2B_2C_2\overline{A}_2\overline{B}_2\overline{C}_2G_3\overline{G}_2)$, and by T^3 the image is composed of the curve $C_{78}(A_{24}B_{24}C_{24}\overline{A}_{30}\overline{B}_{30}\overline{C}_{30}G_{18}\overline{G}_{36})$ and the point G, so that the image of $\overline{G}'\equiv G$ is the same point.

Also the point $\overline{A}'=B$ goes into a curve $C_5(A_1B_2C_2\overline{A}_2\overline{B}_2\overline{C}_2G_1\overline{G}_2)$ by T, and by T^2 the residual image is $C_4(A_2B_2C_1\overline{A}_1\overline{B}_1\overline{C}_1G_2\overline{G}_1)$. Under T^3 the image is the curve $C_{52}(A_{16}B_{16}C_{16}\overline{A}_{20}\overline{B}_{20}\overline{C}_{20}G_{12}\overline{G}_{24})$ and the point $\overline{A}'=B$, so that again T^3 is an identity.

CORNELL UNIVERSITY, *June*, 1917.

planes of the cubics are right lines. The surface is thus a ruled quartic of genus 1, and the system γ is cut from it by an algebraic system of ∞^1 tangent planes.

8. If the given surface is a quintic, the residual section by a plane or hyperplane containing a cubic of γ is a conic. If a generic residual conic is not composite, the surface is generated by a non-rational pencil of conics. Such a surface* has three concurrent double lines and a tacnode. The cubics γ lie in the planes which pass through the tacnode and are tangent to the surface. This surface is normal in three dimensions.

If a generic residual conic is composite, the quintic is a ruled surface of genus 1 belonging to a space of four dimensions or it is the projection of such a surface. The cubic curves on a given ruled quintic surface of genus 1 belonging to S_4 constitute the residual intersections of the surface with the S_3 defined by two generators. Since these cubics constitute a system ∞^1, and intersect in one variable point, they constitute a system γ.

9. *No ruled surface that contains a system γ of cubics is of order greater than* 5. For the generators of such a surface set up, between the points of two generic cubics of γ, a $(1, 1)$ correspondence in which at least one point, common to the two cubics, is self-corresponding since it is not multiple on the surface. The order of the surface defined by such a correspondence does not exceed 5.

10. In the remaining cases, we shall transform the given surface birationally into a ruled quintic surface ϕ of genus one, belonging to S_4. We point out here, for use in this connection, some properties of such surfaces ϕ.

11. *Precisely* $\frac{5x(x+1)}{2}$ *independent linear conditions must be satisfied by the coefficients in the equation of an hypersurface H_1^x of order x, in S_4, in order that it contain a given surface ϕ.*

This theorem is true for $x=1$, since ϕ does not lie in an S_3. We assume it true for all orders less than the given order x. Since the rectilinear generators of ϕ intersect each cubic of γ, a necessary and sufficient condition that H^x contains ϕ is that it contains $x+1$ generic cubics $C_1, C_2, \ldots, C_{x+1}$ of γ. An H^x contains the elliptic cubic C_1 if it contains $3x$ generic points on C_1. It then contains C_2 if it contains $3x-1$ generic points on C_2, etc.

These $3x+3x-1+\ldots+2x=\frac{5x(x+1)}{2}$ linear conditions on the coefficients in the equation of an H^x are independent. For, there exists an H^x which

*Cf. the author, American Journal of Mathematics, Vol. XXX (1908), pp. 115–116.

contains $C_1, C_2, \ldots, C_{j-1}$ (where j has any of the values $1, 2, \ldots, x+1$) and which also contains $3x-j$ generic points $P_j, P_{j+1}, \ldots, P_{3x-1}$ on C_j but which does not contain C_j. There exists, in fact, an hypersurface H_1^{x-1}, of order $x-1$, which contains $C_1, C_2, \ldots, C_{j-2}$ and $P_j, P_{j+1}, \ldots, P_{3x-4}, P_{3x-3}$ and an hyperplane H_1' that contains C_{j-1} and P_{3x-2}. Similarly, there exists a second hypersurface H_2^{x-1}, that contains $C_1, C_2, \ldots, C_{j-2}$ and $P_j, P_{j+1}, \ldots, P_{3x-4}$, P_{3x-2} and a second hyperplane H_2' that contains C_{j-1} and P_{3x-3}. Since the points P are generic, the composite hypersurface $H_1^{x-1}H_1'$ intersects C_j in two points that do not lie on $H_2^{x-1}H_2'$. Thus, no hypersurface of the pencil $\lambda_1 H_1^{x-1}H_1' + \lambda_2 H_2^{x-1}H_2' = 0$ contains C_j. All the hypersurfaces of this pencil contain $C_1, C_2, \ldots, C_{j-1}$ and $P_j, P_{j+1}, \ldots, P_{3x-2}$. One hypersurface of the pencil also contains P_{3x-1}.

12. Any curve on ϕ that intersects a generic generator in x points and a generic cubic of γ in y points is of order $y+2x$, since it intersects the S_3 containing two generic generators and a cubic of γ in $y+2x$ points.

13. *For no curve on ϕ can x be greater than $2y$.* For, the order of such a curve would be $n=y+2x<\frac{5x}{2}$. Since $\frac{5x(x-1)}{2}-n(x-1)>0$, such a curve would lie on an H^{x-1} which does not contain ϕ but has x points in common with each generator.

14. Any curve on ϕ for which $2y=x$ lies on an H^x, since

$$\frac{5x(x+1)}{2}-\frac{5x}{2}\cdot x>0.$$

The residual intersection consists of $5x-\frac{5x}{2}=\frac{5x}{2}$ generators of ϕ, since it does not intersect a generic generator.

There are three curves on ϕ for which $x=2, y=1$. For, put the cubics of γ into $(1,1)$ correspondence with the points of a plane cubic curve C. Then each point of ϕ corresponds to the pair of points on C defined by the two cubics of γ through P. Then the points of a rectilinear generator on ϕ correspond to the pairs of a linear series g_2^1 on C and the points of a cubic of γ to the pairs for which one point is fixed. There are three irrational involutions of order 2 on C defined by the three cubics of which C is the Hessian-Steinerian. Two pairs of such an involution belong to a given g_2^1 and one pair contains a given point P. The points on ϕ defined by these three involutions thus constitute three quintic curves, D_1, D_2, D_3 of genus 1, for which $x=2, y=1$.

We shall show (Art. 35) that there is a rational pencil of curves on ϕ for which $x=4, y=2$. Each of the curves D_1, D_2, D_3 counted twice is a curve of the

pencil. Every curve on ϕ for which $x=2y$ degenerates into curves of this pencil together, possibly, with one or more of the curves D. For let C be such a curve. If C is composite, each component intersects the generators in twice as many points as it does the cubics of γ so that C lies on a proper or composite H^x and forms, with $\frac{5x}{2}$ generators, the complete intersection of H^x with ϕ. Let P be a generic point on C and let C' be the curve of order 10 of the above pencil through P. Then C' has in common with H^x the point P and $10x$ points on the generators common to ϕ and H^x. Hence C' (or a component of it, if P lies on a quintic D) lies on H^x. It forms a component of C since, otherwise, H^x would have more than x points in common with a generic generator of ϕ.

III. The Surface is Non-rational and of Order not Less than 6.

15. Since the order of the surface exceeds 5, the surface is not ruled (Art. 9). The system γ is of order unity (Art. 5, footnote) and index 2.* It is thus representable by pairs of points on a plane cubic curve (cf. Art. 14) and is of genus $p_g=0$, $p_n=-1$.

16. Let the given surface F be projected, if necessary, into a surface F', of order m, in S_3. Then the cubics on F' that constitute the projections of the cubics of γ on F intersect the adjoints to F' of order $m-3$ in just two points which are not multiple on F'. Suppose, in fact, that the cubics do not have a fixed point in common. Since consecutive cubics intersect, the plane of a generic cubic C touches F' at only two of the intersections of C with the plane of the consecutive cubic. The remaining $3m-11$ points common to C and the residual section of its plane with F' lie on the multiple curve and thus on the adjoints of order $m-3$. Similarly, if the cubics have a fixed point P in common, the plane of C touches F' at one point on C. The remaining intersections of C with the residual curve, except one at P, are common to the adjoints of order $m-3$.

17. *The adjoints of order $m-3$ constitute a linear system of at most two dimensions.* Suppose, in fact, that they constituted a linear system of $r>2$ dimensions. Since the cubics are not rational, they determine a linear series g_2^1 on each cubic of γ. Then the ∞^{r-3} adjoints of order $m-3$ through two generic points of one cubic and one generic point of a second generic cubic would contain all the cubics and would thus contain F'. But this is impossible, since the order of the adjoints is less than that of F'.

* Castelnuovo–Enriques, *Mathematische Annalen*, Vol. XLVIII, p. 314.

18. *The genus of the plane sections of F' does not exceed* 4. Let Π be the genus of a generic plane section of F' and let r be the dimension of the linear system of adjoints of order $m-3$. Then*

$$r=\Pi-1-(p_g-p_n) \quad \text{or} \quad \Pi=r+1+p_g-p_n.$$

But $r\leq 2$, $p_g=0$, $p_n=-1$, so that $\Pi\leq 4$. Moreover, $\Pi\geq 3$, since the surface is not ruled.†

19. If $\Pi=3$, it is known‡ that the surface contains an irrational pencil of conics. The quartics cut from such a surface F' by the pencil of adjoints of order $m-3$ degenerate into pairs of conics.

If $\Pi=4$, then $r=2$ so that a pencil of adjoints of order $m-3$ pass through a generic point P on F'. All the adjoints of this pencil pass through two points P_1 and P_2 fixed by P on the cubics C_1 and C_2, respectively, through P. In case P_1 and P_2 lie on a cubic C of γ, they are corresponding points in the g_2^1 defined on C by the adjoints. Then the adjoint surface that contains C contains all the pairs of the g_2^1 on C and thus contains a rational cubic curve (locus of P) which forms, with C, the variable sextic of intersection of the adjoint with F'. In case P_1 and P_2 do not lie on the same cubic, let C_1' and C_2' be cubics of γ through P_1 and P_2, respectively. The pencil of adjoints through P_1 intersect C_1' in P_1 and in a second fixed point distinct from its intersection with C_2. It follows that the adjoint surface that contains C_2 has C_1' for its residual intersection and, similarly, that the one containing C_1 has C_2' for its residual intersection.

20. Let the system γ of cubics on a surface F, belonging to three or more dimensions, be put on $(1, 1)$ correspondence with the system of cubic curves on a ruled quintic surface ϕ, of genus 1, belonging to S_4. Then a $(1, 1)$ correspondence is set up between F and ϕ by taking two points as corresponding when they lie at the intersection of corresponding pairs of cubics.

21. Let $x>1$ be the order of the rational curves on F. Then the plane or hyperplane sections of F transform into a linear system of curves on ϕ that are of order $3+2x$ since they intersect each cubic in three points and each generator in x points and thus intersect any S_3 which contains a cubic and two generators in $3+2x$ points. Every such curve lies on an H^x, since $\dfrac{5x(x+1)}{2}$ $-x(2x+3)>0$ when $x>1$. The residual intersection of the H^x with ϕ consists of $3(x-1)$ generators. The plane or hyperplane sections of F thus correspond to sections of ϕ by a linear system of H^x through $3(x-1)$ fixed generators.

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* Picard et Simart, "Théorie des Fonctions Algébriques," Vol. II, p. 489.

† Castelnuovo–Enriques, *Mathematische Annalen*, Vol. XLVIII (1897), p. 308.

‡ Scorza, *Annali di Matematica*, Ser. 3, Vol. XVI, p. 255, *et seq.*

The order of the rational curves on F does not exceed 4. For, since $y=3$, we have at once $x \leq 5$ (Arts. 13, 14). But the H^5 through twelve generators define on ϕ a linear system of only two dimensions so that $x \leq 4$.

a. *The Rational Curves on F are Conics.*

22. Since $x=2$ and $y=3$, the curves on ϕ corresponding to the plane or hyperplane sections of F are of order 7 and constitute the residual intersection of ϕ with a system of H^2 through three fixed lines. The complete linear system to which they belong is of order 8 and dimension 5. Hence the surface F is of order 8 and belongs to a space of five dimensions, or it is the projection of such a surface. This surface is the "first type" discussed by Scorza in an article entitled "Le superficie a curve sezioni di genere 3" in the *Annali di Matematica*, Ser. 3, Vol. XVII (1910), p. 320.

b. *The Rational Curves on F are Cubics.*

23. To the system of plane or hyperplane sections of F corresponds a linear system of curves of order 9 cut from ϕ by H^3 through six fixed lines. The complete linear system defined by these curves is of order 9 and dimension 5. Hence, *the given surface F is of order* 9 *and belongs to a space of five dimensions, or it is the projection of such a surface.* A generic hyperplane section is of genus 4 (Arts. 18 and 19).

24. Let F belong to S_5. Since two generic cubics of γ intersect, they lie in an S_4. The residual intersection of this S_4 with F is a cubic of γ since it corresponds to a cubic on ϕ. Since any curve on F intersects three such cubics of γ in the same number of points, we deduce that: *the order of any curve on F is equal to $3y$,* where y is the number of its intersections with a generic curve of γ.

25. There are three types of cubic curves on F, defined by the number x, of their intersections with the rational curves on F. If $x=0$, the curves are the rational cubics; if $x=1$, they are the curves of γ; if $x=2$, there are just three curves D'_1, D'_2, D'_3. They are of genus 1, and correspond to the three quintics on ϕ (Art. 14) that intersect the generators in two points. The planes of two such cubics D' do not intersect. Otherwise, they would lie in an S_4 which would have four points in common with each rational cubic on F.

26. The rational cubics define a non-rational involution of order 2 on each cubic D'. The lines joining corresponding points of such an involution envelope a curve of class 3 in the plane of D'. It follows that the S_3 defined by the rational cubics on F generate an hypersurface of order 6, since an S_3 which contains a generic line l_1 in the plane of D'_1 and l_2, in the plane of D'_2, intersects

the locus of the S_3 defined by the rational cubics in a ruled surface which has l_1 and l_2 as triple directrices and is thus of order 6. The point in which the S_3 intersects the plane of D'_3 is a triple point on the sextic. Hence the generator through that point is a triple generator on the ruled surface. It follows that the cubic threespread formed by the lines that intersect the planes of D'_1, D'_2 and D'_3 is a triple threespread on the sextic hypersurface. The hypersurface is of genus 1, and has no other multiple points.

27. Let S be the threespace of a generic rational cubic C on F. The plane of a cubic D' has a line in common with S and determines, with S, an S_4 whose residual intersection with F is a cubic of γ whose plane has a line in common with S. Conversely, if Γ is a cubic of γ whose plane has a line in common with S, then Γ, C and a cubic D' lie in an S_4. Each of the two intersections of Γ with S that do not lie on C lies on a cubic D'. Moreover, the threespread generated by the planes of the cubics γ is of order 9, since it intersects S in three right lines and in the cubic C which is counted twice since two cubics γ pass through each point of C.

28. The S_4 that intersect F in three cubics of γ (Art. 24) define a g_3^2 on the system γ. Let these S_4 be put in (1, 1) correspondence with the points of a plane in such a way that the S_4 corresponding to the points on a line define a g_3^1 on γ. Each cubic of γ belongs to one group of such a g_3^1, so that each point of F, and thus each point of S_5, lies in two of the S_4 define by the points of a line. Then the $\infty^1 S_4$ that pass through a given point correspond to the points of a conic or: the locus of the S_4 that intersect F in three cubics of γ is the dual of a surface of Veronese.

29. The sextics on F intersect the cubics γ in two points (Art. 24). Those that intersect the rational cubics once, correspond to the quartic curves on ϕ. They constitute ∞^1 bundles, intersect in three points and are of genus 1. Three of these bundles constitute the residual intersections of the bundles of S_4 containing a cubic D'. A fourth bundle constitutes the adjoint sextics (Art. 19) to the hyperplane sections of F.

The sextics that intersect the rational cubics twice are of genus 2, intersect in four points and constitute the ∞^1 bundles forming the residual intersections of the S_4 that contain a cubic γ. Those that intersect the rational cubics in three points are of genus 2, intersect in three points and constitute the ∞^1 pencils residual to the rational cubics. Those that intersect the rational cubics in four points are of genus 1 and constitute a pencil (Art. 14).

30. The projection of F on an S_3 from a generic line l in the plane of a cubic C of γ is a sextic surface with a tacnode at the intersection of the plane

of C with S_3. The projections of the cubics of γ pass through the tacnode and lie in pairs in the planes tangent to a quadric cone. The rational cubics also pass through the tacnode. Three of them are nodal, and coplanar with the projections of the cubics D'. The surface has three coplanar double lines, projections of cubics of γ that intersect l. The points of intersection of these double lines are triple points on the surface. The residual nodal cubic passes through these triple points.

31. The projection of F on an S_3 from a generic line l in the plane of a cubic D' is a sextic surface with a triple point at the intersection of the plane of D' with S_3. The rational cubics have a node at this triple point. Their planes envelope a cone of class 3. These planes also contain the cubics γ. The six cubics of γ that intersect l project into double lines forming the six edges of a tetrahedron.

c. *The Rational Curves on F are Quartics.*

32. To the system of plane or hyperplane sections of F corresponds a linear system of curves of order 11 cut from ϕ by H^4 through nine fixed lines. The complete linear system defined on ϕ by these curves is of order 8 and dimension 4. Hence, *the given surface F is of order 8 and belongs to four dimensions, or it is the projection of such a surface.* A generic hyperplane section is of genus 4 (Arts. 18 and 19).

33. Let F belong to an S_4. An H^4 that defines a curve on ϕ corresponding to an hyperplane section of F, has eighteen of its twenty intersections with a quintic D (Art. 14) fixed on the nine fundamental lines. These H^4 thus determine a g_2^1 on each quintic D so that *the surface F has three double lines* d_1, d_2, d_3. No two of these double lines can intersect. Otherwise, each S_3 of the pencil containing them would have for residual intersection with F a rational quartic (since it corresponds to a right line on ϕ) from which it would follow that F is rational.

34. The rectilinear generators of ϕ determine on each quintic D a non-rational involution of order 2 which has two pairs of points in common with the g_2^1 defined by the curves corresponding to the hyperplane sections of F. It follows that two rational quartics on F have a double point on each double line.

Let S be the threespace defined by d_2 and d_3. Its residual intersection with F is a rational quartic C_1. The two intersections of C_1 with d_1 coincide at the intersection of d_1 with S. Hence C_1 has a double point on d_1. Similarly, d_1 and d_3 define a quartic C_2 with a double point on d_2 and d_1, d_2 a quartic C_3 with a double point on d_3. The three double points are collinear.

Let C be the rational quartic, other than C_1, that has a double point on d_1. Then, since C does not lie in the S_4 containing d_2 and d_3, C has a double point on a second double line d_2 and lies in a plane. This plane can not contain d_3. Otherwise, the S_3 through it would define a rational pencil of conics on F. Hence the third double point of C lies on d_3.

35. The residual intersections with F of the S_3 through the plane of C constitute a rational pencil of quartics of genus 1 which intersect the rational quartics in four points and the cubics of γ in two points. No two quartics of this pencil intersect. The curves on ϕ corresponding to these quartics are of order 10. They intersect the generators in four points and the cubics of γ twice.

36. No rational quartic on F, other than C, C_1, C_2, C_3, lies in an S_3. Otherwise, let C' be such a quartic. It has a double point on at least two double lines (since it can not lie in the S_3 defined by two double lines) and lies in a plane. The residual intersections of the S_3 containing this plane constitute a pencil of quartics, distinct from that of Art. 35, which intersect the rational curves four times and the cubics of γ twice. This is impossible (Art. 14).

37. Let Γ be any cubic of γ. The plane of Γ intersects a given double line d_1 and defines with it an S_3 whose residual intersection with F is a second cubic Γ' of γ. The points in which d_2 and d_3 intersect S_3 lie on Γ and Γ' and thus on the line of intersection of their planes. It follows that each cubic of γ is intersected in two points (which lie on two of the double lines) by each of three other cubics of γ.

Let S be the threespace defined by d_1 and d_2. Four cubics of γ pass through a generic point of d_1. The lines in which the planes of these cubics intersect S coincide in pairs and intersect d_2. These lines thus establish a (2, 2) correspondence between the points of d_1 and d_2 and generate a ruled quartic surface. The generators of this surface are bisecants of the rational curve C_3 (Art. 34). The common secant line of d_1, d_2, d_3 is a double generator of the ruled quartic.

The hypersurface generated by the planes of γ is of order 8, since its section by the S_3 of two double lines is a ruled quartic counted twice. This octavic hypersurface has F and the three ruled quartic surfaces defined by pairs of the lines d_1, d_2, d_3 as double surfaces. Since it is of genus 1, it has no other multiple points.

38. *The surface F forms, with the plane of the rational plane quartic on it, the basis surface of a pencil of cubic hypersurfaces.* For, the H^3 that contain the three double lines and six generic cubics of γ contain all the cubics of γ and

thus contain F. That an H^3 contain four points on each double line is twelve conditions, that it further contain six generic cubics of γ is twenty-one more conditions on the thirty-five homogeneous coefficients in the equation of the hypersurface. There thus exists a pencil of such H^3. The plane of the rational quartic C (Art. 34) clearly lies on all these H^3 since every line in it has four points in common with each H^3.

39. Let a curve of order n on F intersect a generic rational quartic in x points and a generic cubic of γ in y points. Then

$$4y=x+n.$$

For, the corresponding curve on ϕ is of order $y+2x$. Of its $4(y+2x)$ intersections with the H^4 that defines a generic hyperplane section of F, $9x$ lie on the nine fundamental lines (Art. 32). The remaining $4(y+2x)-9x=n$ define the intersections of the given curve with the given hyperplane. Since $y\leq 2x$, we have $x\leq n$. The equality sign holds (for non-composite curves) only for $x=2$ and $x=4$ (Art. 14).

If $n=2$, we have $x=2$, $y=1$. The curves are the three double lines.

If $n=3$, we have $x=1$, $y=1$. The curves are the cubics of γ.

If $n=4$ and $x=0$, $y=1$, the curves are the rational quartics.

If $n=4$ and $x=4$, $y=2$, the curves are the pencil of residual quartics in the S_3 through the plane of the rational quartic C (Art. 34).

If $n=5$, we have $x=3$, $y=2$. There are ∞^1 pencils of these curves. They form the residual intersections of the S_3 that contain a cubic of γ. They are of genus 2 and intersect in three points.

If $n=6$, we have $x=2$, $y=2$. There are ∞^1 bundles of such curves. They are of genus 2 and intersect in four points. In each pencil in any bundle, there are two which break up into pairs of cubics of γ. One such bundle constitutes the bundle of adjoint sextics to the hyperplane sections of F (Art. 19).

If $n=7$ and $x=1$, $y=2$, we have ∞^1 bundles of curves of genus 1 that intersect in three points. If $x=5$, $y=3$, we have ∞^1 bundles of curves of genus 3 that intersect in five points. All the curves of a bundle intersect each double line in a fixed point.

40. The projection of F on an S_3 from a generic point on a double line d is a sextic surface having a tacnode at the intersection of d with S_3. The tacnodal tangent plane contains the projection of the rational plane quartic and two right lines. The cubics γ lie in pairs in the tangent planes to a quadric cone with vertex at the tacnode. The rational quartics have a double point at the tacnode. The double curve is composed of six right lines forming the edges of a tetrahedron.

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An Isoperimetric Problem with Variable End-Points.

By Archibald Shepard Merrill.

Introduction.

The object of this paper is to give a complete discussion of the necessary and sufficient conditions for a maximum (minimum) for a type of problems in the Calculus of Variations which are closely related to the usual isoperimetric problems, and in which both end-points are allowed to vary along a given fixed curve. We suppose that we have given the fixed curve L and a certain arc E_{12}

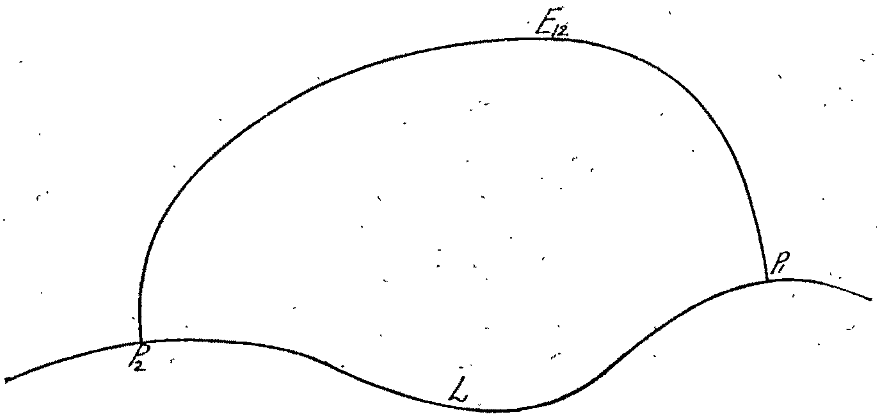

Fig. 1.

joining two points P_1 and P_2 of L. The problem is to determine the properties which the curve E_{12} must have in order that the integral of a given function $F(x, y, x', y')$ along L from P_2 to P_1, then along E_{12} from P_1 to P_2 shall be a maximum (minimum), while the integral of a second function $G(x, y, x', y')$ along E_{12} has a prescribed value. Thus the function J to be maximized (minimized) is a sum of two integrals:

$$J = \int_{L_{21}} F(x, y, x', y')\, dx + \int_{E_{12}} F(x, y, x', y')\, dt,$$

while the integral

$$K = \int_{E_{12}} G(x, y, x', y')\, dt$$

is to remain fixed in value.

A familiar application of this type of problem is the well-known Problem of Dido. In this application the area included by the arc P_1P_2 of E_{12} and the arc P_2P_1 of L is to be a maximum, while the arc P_1P_2 of E_{12} is to have a preassigned length.

For the problem at hand certain conditions of the corresponding isoperimetric problem with fixed end-points must hold, and these are already well known, viz., the Euler, Weierstrass, Legendre and Jacobi necessary conditions.* The transversality condition is also readily obtainable, and has been deduced for some special cases of the problem here discussed.† In the present paper a new necessary condition, corresponding to the Jacobi condition in other problems in the Calculus of Variations, will be deduced and discussed both for the case of one end-point variable and for the case of both end-points variable. In obtaining this we make use of the derivatives of the "extremal integral" for any isoperimetric problem, and Section 2 is given over entirely to the computation of these derivatives. Conditions which are sufficient for a maximum of J when K is fixed are readily obtained with the help of a theorem proved by Hahn. A geometric interpretation of the new condition is given in Section 5. Finally in Section 6, as an application of the general theory developed, a discussion of the above-mentioned Problem of Dido is given.

§ 1. *Conditions Deducible from Known Results.*

Consider a fixed curve

$$x=\tilde{x}(\varkappa), \quad y=\tilde{y}(\varkappa) \tag{L}$$

not intersecting itself, and two points $P_1(\varkappa=\varkappa_1)$ and $P_2(\varkappa=\varkappa_2)$ on L with $\varkappa_2<\varkappa_1$. Let E_{12} be an arc

$$x=\phi(t), \quad y=\psi(t), \quad t_1 \leqq t \leqq t_2, \tag{E}$$

cutting L at $P_1(t=t_1)$ and $P_2(t=t_2)$. The function to be maximized or minimized is then of the form

$$J=\int_{\varkappa_2}^{\varkappa_1} F(\tilde{x}, \tilde{y}, x', y')\, d\varkappa + \int_{t_1}^{t_2} F(\phi, \psi, \phi', \psi')\, dt,$$

while the integral

$$K=\int_{t_1}^{t_2} G(\phi, \psi, \phi', \psi')\, dt$$

is to remain constant in value.

For simplicity the following discussion will be restricted to the determination of a maximum for J. Consider the totality of arcs whose end-points

* See Bolza, "Vorlesungen über Variationsrechnung, Chapter X.

† See Bolza, *loc. cit.*, p. 520.

lie upon L. In this class there is a sub-class $\mathfrak{M}$ of arcs which give the integral K a fixed value k. The problem is then to find necessary and sufficient conditions that a particular arc E_{12} of $\mathfrak{M}$, intersecting arc L at $P_1(x=x_1)$ and $P_2(x=x_2)$, shall give to J a larger value than any other arc of $\mathfrak{M}$ in a certain neighborhood of arc E_{12}.

It is presupposed that the arc L is regular* in a neighborhood of the values $x_2 \leqq x \leqq x_1$. The class $\mathfrak{M}$ is further restricted to contain only regular arcs, and in particular the arc E_{12} whose maximizing properties are to be investigated is assumed to be of class C'''.† It is also assumed that E is not an extremal for the integral K.

The function G is of class C''' for all values (x, y, x', y') for which $(x', y') \neq (0, 0)$ and (x, y) is in a neighborhood of E_{12}, while F is of the same class for $(x', y') \neq (0, 0)$ and (x, y) in a neighborhood of those on $L_{12}+E_{12}$. Both these functions satisfy the usual homogeneity conditions

$$F(x, y, kx', ky') = kF(x, y, x', y'),$$
$$G(x, y, kx', ky') = kG(x, y, x', y')$$

in these neighborhoods for every $k>0$.

The necessary conditions that $J(E)$ be a maximum with respect to all curves of $\mathfrak{M}$ with the same end-points, and keeping K fixed, must be fulfilled. Hence we have at once the usual Euler, Weierstrass, Legendre and Jacobi conditions referred to above. These may be stated as follows:

I. *Euler Condition.*—The curve E must satisfy for a certain constant value λ the Euler differential equations

$$H_x - \frac{d}{dt}H_{x'} = 0, \quad H_y - \frac{d}{dt}H_{y'} = 0, \tag{1}$$

where $H = F + \lambda G$. Such curves will, as usual, be called extremals.

II. *Weierstrass Condition.*—The Weierstrass E function

$$E(x, y, p, q, x', y'; \lambda) = H(x, y, x', y'; \lambda) - x'H_{x'}(x, y, p, q; \lambda) - y'H_{y'}(x, y, p, q; \lambda)$$

must be greater than or equal to zero for all $(x, y, p, q; x', y')$, such that (x, y, p, q) belongs to a point of E_{12} while (x', y') is different from $(0, 0)$.

III. *Legendre Condition.*—Along the arc E_{12}, $H_1 \leqq 0$, where

$$H_1 = \frac{H_{x'x'}}{y'^2} = -\frac{H_{x'y'}}{x'y'} = \frac{H_{y'y'}}{x'^2}.$$

* An arc is said to be regular when it is continuous and consists of a finite number of arcs each of which has a well-defined and continuously turning tangent.

† See Bolza, *loc. cit.*, p. 13.

IV. Finally E_{12} must satisfy the *Jacobi Condition* for fixed end-points; that is, the extremal arc E_{12} must not contain in its interior either the point P'_1 conjugate to P_1, or the point P'_2 conjugate to P_2.

In the transversality condition there is a departure from the result obtained by Bolza* in a closely related problem. We proceed to its determination, however, in an analogous manner. Consider one end-point, say P_1, fixed. It is possible to set up in the usual way† a one-parameter family of variation curves

$$x=\phi(t, \varkappa), \quad y=\psi(t, \varkappa), \quad t_1 \leqq t \leqq t_2,$$

which have the following properties. For $\varkappa=\varkappa_2$, $t_1 \leqq t \leqq t_2$, the family contains the arc E_{12}. Furthermore, every arc passes through the point P_1 for $t=t_1$, and intersects the arc L for $t=t_2$, which gives rise to the equations

$$\begin{aligned} \phi(t_1, \varkappa) &= x_1, & \psi(t_1, \varkappa) &= y_1, \\ \phi(t_2, \varkappa) &= x(\varkappa), & \psi(t_2, \varkappa) &= \tilde{y}(\varkappa). \end{aligned}$$

Finally, along each one of these curves, the integral K has the assigned value k. Substituting this family of variation curves in the expression for J we find

$$J(\varkappa) = \int_{\varkappa}^{\varkappa_1} F(x, \tilde{y}, x', \tilde{y}')\, d\varkappa + \int_{t_1}^{t_2} F(\phi, \psi, \phi', \psi')\, dt.$$

Following the procedure of Bolza we have the condition that at the point P_2,

$$F(\tilde{x}, \tilde{y}, \tilde{x}', y') - H_{x'}(\phi, \psi, \phi', \psi')\tilde{x}' - H_{y'}(\phi, \psi, \phi', \psi')\tilde{y}' = 0,$$

and by a similar argument at P_1 we have the following result:

V. *Transversality Condition.*—The curve L must cut E_{12} transversally at P_1 and P_2, that is, at both these points the condition

$$F(\tilde{x}, \tilde{y}, x', \tilde{y}') - H_{x'}(\phi, \psi, \phi', \psi')\tilde{x}' - H_{y'}(\phi, \psi, \phi', \psi')\tilde{y}' = 0 \tag{2}$$

must hold.

§ 2. *Derivatives of the Extremal Integral.*

We suppose now that the arc E_{12} satisfies the necessary conditions of the preceding section, and further that the Legendre and Jacobi conditions for fixed end-points hold in the so-called stronger form. This means that $H_1 < 0$ along the arc E_{12}, and that P_1 and P_2 are not conjugate points on E_{12}. As a result of the continuity properties of F and G, and the fact that $H_1 \neq 0$ along E_{12}, it is known‡ that this arc may be imbedded in a family of extremals

$$x=\phi(t, \alpha, \beta, \lambda), \quad y=\psi(t, \alpha, \beta, \lambda) \tag{3}$$

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* *Loc. cit.*, pp. 519, 520. † Bolza, *loc. cit.*, pp. 473, 474. ‡ Bolza, *loc. cit.*, pp. 468 ff.

which contains E_{12} for values $\alpha_0, \beta_0, \lambda_0, t_1 \leqq t \leqq t_2$. The functions $\phi, \phi_t, \psi, \psi_t$ are of the class C' in all their arguments in a neighborhood of these values. The constant λ is the isoperimetric constant for each extremal.

Suppose now that M_1 and M_2 are any two points (x_1, y_1) and (x_2, y_2) sufficiently near to P_1 and P_2, respectively. The equations

$$\left.\begin{array}{c} \phi(\tau_1, \alpha, \beta, \lambda)=x_1, \quad \phi(\tau_2, \alpha, \beta, \lambda)=x_2, \\ \psi(\tau_1, \alpha, \beta, \lambda)=y_1, \quad \psi(\tau_2, \alpha, \beta, \lambda)=y_2, \\ \int_{\tau_1}^{\tau_2} G(\phi, \psi, \phi', \psi')\, dt = k, \end{array}\right\} \tag{4}$$

have the initial solution

$$(\alpha, \beta, \lambda, \tau_1, \tau_2, x_1, y_1, x_2, y_2)=(\alpha_0, \beta_0, \lambda_0, t_1, t_2, x_{10}, y_{10}, x_{20}, y_{20}),$$

where (x_{10}, y_{10}), (x_{20}, y_{20}) are now the coordinates of P_1, P_2, respectively. Furthermore, since the Jacobi condition in its stronger form is satisfied by the arc E_{12}, it follows that the functional determinant of the left members for $\alpha, \beta, \lambda, \tau_1, \tau_2$ is different from zero at this initial solution. It reduces in fact, after suitable transformations, to the determinant $D(t_1, t_2)$* formed for E_{12}, which vanishes only when P_1 and P_2 are conjugate. Hence equations (4) have unique solutions of the form

$$\left.\begin{array}{c} \alpha(x_1, y_1, x_2, y_2), \quad \beta(x_1, y_1, x_2, y_2), \quad \lambda(x_1, y_1, x_2, y_2), \\ \tau_1(x_1, y_1, x_2, y_2), \quad \tau_2(x_1, y_1, x_2, y_2), \end{array}\right\} \tag{5}$$

reducing to $\alpha_0, \beta_0, \lambda_0, t_1, t_2$ for $(x_1, y_1, x_2, y_2)=(x_{10}, y_{10}, x_{20}, y_{20})$, and of class C' near these values. Substituting from these last functions for α, β, λ, in equations (5), we have a family of extremals

$$x=\phi(t, x_1, y_1, x_2, y_2), \quad y=\psi(t, x_1, y_1, x_2, y_2), \tag{6}$$

and two functions, $\tau_1(x_1, y_1, x_2, y_2)$ and $\tau_2(x_1, y_1, x_2, y_2)$ for which the following conditions are then satisfied:

$$\left.\begin{array}{c} \phi(\tau_1, x_1, y_1, x_2, y_2)=x_1, \quad \phi(\tau_2, x_1, y_1, x_2, y_2)=x_2, \\ \psi(\tau_1, x_1, y_1, x_2, y_2)=y_1, \quad \psi(\tau_2, x_1, y_1, x_2, y_2)=y_2, \\ \int_{\tau_1}^{\tau_2} G(\phi, \psi, \phi', \psi')\, dt = k. \end{array}\right\} \tag{7}$$

By differentiating these we find that the following relations hold at the point M_1:

$$\left.\begin{array}{ll} \phi'\tau_{1x_1}+\phi_{x_1}=1, & \psi'\tau_{1x_1}+\psi_{x_1}=0, \\ \phi'\tau_{1y_1}+\phi_{y_1}=0, & \psi'\tau_{1y_1}+\psi_{y_1}=1, \\ \phi'\tau_{1x_2}+\phi_{x_2}=0, & \psi'\tau_{1x_2}+\psi_{x_2}=0, \\ \phi'\tau_{1y_2}+\phi_{y_2}=0, & \psi'\tau_{1y_2}+\psi_{y_2}=0, \end{array}\right\} \tag{8}$$

* Bolza, *loc. cit.*, p. 478.

while at the point M_2,

$$\left.\begin{aligned} &\phi'\tau_{2x_1}+\phi_{x_1}=0, && \psi'\tau_{2x_1}+\psi_{x_1}=0,\\ &\phi'\tau_{2y_1}+\phi_{y_1}=0, && \psi'\tau_{2y_1}+\psi_{y_1}=0,\\ &\phi'\tau_{2x_2}+\phi_{x_2}=1, && \psi'\tau_{2x_2}+\psi_{x_2}=0,\\ &\phi'\tau_{2y_2}+\phi_{y_2}=0, && \psi'\tau_{2y_2}+\psi_{y_2}=1. \end{aligned}\right\} \tag{9}$$

In accordance with the notation and nomenclature of Bolza,* we use the notation

$$I(x_1, y_1, x_2, y_2)=\int_{\tau_1}^{\tau_2}F(\phi, \psi, \phi', \psi')\,dt, \tag{10}$$

and call this expression the extremal integral. We desire to obtain the partial derivatives of the function I with respect to its four arguments. In order to simplify the results we make use of two important functions Ω and Ψ, which will now be introduced.

The problem under consideration may be interpreted as a problem in space by defining a third coordinate z by the equation

$$z=\chi(t, x_1, y_1, x_2, y_2)=\int_{\tau_1}^{t}G(\phi, \psi, \phi', \psi')\,dt, \tag{11}$$

where ϕ, ψ are of the form given in (6). By differentiation of (11) with respect to t and α, where α is an arbitrarily selected one of the elements x_1, y_1, x_2, y_2, we obtain

$$-\chi_t+G(\phi, \psi, \phi', \psi')=0, \quad \chi_\alpha=\int_{\tau_1}^{t}(G_x\phi_\alpha+G_y\psi_\alpha+G_{x'}\phi'_\alpha+G_{y'}\psi'_\alpha)-G\,|^1\tau_{1\alpha}. \tag{12}$$

If h is defined by the equation

$$h=F+\lambda(G-z')=H-\lambda z',$$

then the equations

$$h_x-\frac{d}{dt}h_{x'}=0, \quad h_y-\frac{d}{dt}h_{y'}=0, \quad h_z-\frac{d}{dt}h_{z'}=0 \tag{13}$$

are satisfied along any extremal arc, and might be called the Euler equations of the space problem.

We now set up the function Ω by means of the following equation:

$$\begin{aligned} 2\Omega=&\begin{pmatrix}H_{xx}H_{xy}\\ H_{yx}H_{yy}\end{pmatrix}(\xi, \eta)(\xi, \eta)+2\begin{pmatrix}H_{x'x}H_{x'y}\\ H_{y'x}H_{y'y}\end{pmatrix}(\xi, \eta)(\xi', \eta')\\ &+\begin{pmatrix}H_{x'x'}H_{x'y'}\\ H_{y'x'}H_{y'y'}\end{pmatrix}(\xi', \eta')(\xi', \eta')+2\mu(G_x\xi+G_y\eta+G_{x'}\xi'+G_{y'}\eta'-\zeta'), \end{aligned} \tag{14}$$

in which the notation is explained by the equation

$$\begin{pmatrix}A_{11}A_{12}\\ A_{21}A_{22}\end{pmatrix}(x_1, y_1)(x_2, y_2)=(A_{11}x_1+A_{12}y_1)x_2+(A_{21}x_1+A_{22}y_1)y_2.$$

* *Loc. cit.*, pp. 308 ff.

Then the equations analogous to those of Jacobi for other problems in the Calculus of Variations are the following:

$$\left.\begin{aligned}
\Omega_\xi-\frac{d}{dt}\Omega_{\xi'}&\equiv H_{xx}\xi+H_{xy}\eta+H_{xx'}\xi'+H_{xy'}\eta'+G_x\mu\\
&\qquad-\frac{d}{dt}\left(H_{x'x}\xi+H_{x'y}\eta+H_{x'x'}\xi'+H_{x'y'}\eta'+G_{x'}\mu\right)=0,\\
\Omega_\eta-\frac{d}{dt}\Omega_{\eta'}&\equiv H_{yx}\xi+H_{yy}\eta+H_{yx'}\xi'+H_{yy'}\eta'+G_y\mu\\
&\qquad-\frac{d}{dt}\left(H_{y'x}\xi+H_{y'y}\eta+H_{y'x'}\xi'+H_{y'y'}\eta'+G_{y'}\mu\right)=0,\\
\Omega_\zeta-\frac{d}{dt}\Omega_{\zeta'}&\equiv\quad\frac{d}{dt}\mu=0,\\
\Omega_\mu-\frac{d}{dt}\Omega_{\mu'}&\equiv G_x\xi+G_y\eta+G_{x'}\xi'+G_{y'}\eta'-\zeta'=0.
\end{aligned}\right\}\tag{15}$$

These equations are satisfied by the functions

$$(\xi,\eta,\zeta,\mu)=(\phi_a,\psi_a,\chi_a,\lambda_a)\tag{16}$$

obtained from (5), (6) and (12), where α stands for any one of x_1, y_1, x_2, y_2. This is proved by substituting the functions (6) in the Euler equations (1) and differentiating the resulting identities with respect to α, and by differentiating (12) for t.

The expressions for the values of χ_a at the points M_1 and M_2 will be useful in later simplifications, and will be computed now. The last equation of (7) is satisfied by the functions (6), and hence by differentiation we obtain

$$\int_{\tau_1}^{\tau_2}(G_x\phi_a+G_y\psi_a+G_{x'}\phi'_a+G_{y'}\psi'_a)\,dt+G\,|^2t_{1a}-G\,|^1t_{1a}=0,$$

that is, from (12),

$$\chi_a(\tau_2,x_1,y_1,x_2,y_2)=-G\,|^1t_{1a},\tag{17}$$

where it is to be remembered that α is now not one of the constants in (3), but one of the variables x_1, y_1, x_2, y_2. By direct computation from the expression (12) for χ_a, we obtain

$$\chi_a(\tau_2,x_1,y_1,x_2,y_2)=-G\,|^1\tau_{1a}.\tag{18}$$

Since Ω is a quadratic form in $\xi, \eta, \zeta, \mu, \xi', \eta', \zeta', \mu'$, we have the relations

$$\begin{aligned}
\Sigma(\xi\,\Omega_\xi+\xi'\Omega_{\xi'})&=2\Omega,\\
\Sigma(\xi_1\Omega_{\xi_2}+\xi'_1\Omega_{\xi'_2})&=\Sigma(\xi_2\Omega_{\xi_1}+\xi'_2\Omega_{\xi'_1}),
\end{aligned}$$

where Σ denotes summation over the four elements ξ, η, ζ, μ, and the notation Ω_{ξ_2}, for example, denotes the function obtained by differentiating Ω with

respect to ξ, and then replacing ξ by ξ_2, η by η_2, etc. It follows from the second of these that

$$\Sigma\left\{\xi_1\left(\Omega_{\xi_2}-\frac{d}{dt}\Omega_{\xi'_2}\right)-\xi_2\left(\Omega_{\xi_1}-\frac{d}{dt}\Omega_{\xi'_1}\right)\right\}=-\frac{d}{dt}\Psi\begin{pmatrix}\xi_1, \eta_1, \zeta_1, \mu_1, \xi'_1, \eta'_1, \zeta'_1, \mu'_1\\ \xi_2, \eta_2, \zeta_2, \mu_2, \xi'_2, \eta'_2, \zeta'_2, \mu'_2\end{pmatrix},$$

where

$$\Psi=\Sigma(\xi_1\Omega_{\xi'_2}-\xi_2\Omega_{\xi'_1}).$$

We see, therefore, that the function Ψ is a constant if $(\xi_1, \eta_1, \zeta_1, \mu_1)$ and $(\xi_2, \eta_2, \zeta_2, \mu_2)$ are both solutions of equations (15).

We may now proceed to the computation of the derivatives of the extremal integral. Let α, β represent any two (or possibly both the same one) of $x_1, y_1; x_2, y_2$. By differentiating (10) we obtain

$$I_\alpha=\int_{\tau_1}^{\tau_2}(F_x\phi_\alpha+F_y\psi_\alpha+F_{x'}\phi'_\alpha+F_{y'}\psi'_\alpha)\,dt+F\tau_\alpha\Big|_1^2. \qquad (19)$$

From the isoperimetric condition we have

$$0=\lambda\int_{\tau_1}^{\tau_2}(G_x\phi_\alpha+G_y\psi_\alpha+G_{x'}\phi'_\alpha+G_{y'}\psi'_\alpha)\,dt+\lambda G\tau_\alpha\Big|_1^2.$$

Then by adding, performing the Lagrangian partial differentiation, applying the Euler equation (1), and using the well-known homogeneity relation

$$H=x'H_{x'}+y'H_{y'},$$

we obtain

$$I_\alpha=H_{x'}(\phi'\tau_\alpha+\phi_\alpha)+H_{y'}(\psi'\tau_\alpha+\psi_\alpha)\Big|_1^2. \qquad (20)$$

Similarly,

$$I_\beta=H_{x'}(\phi'\tau_\beta+\phi_\beta)+H_{y'}(\psi'\tau_\beta+\psi_\beta)\Big|_1^2. \qquad (21)$$

Differentiating I_α with respect to β we have

$$\begin{aligned}I_{\alpha\beta}=(\phi'\tau_\alpha+\phi_\alpha)&\left[\tau_\beta\frac{d}{dt}H_{x'}+\Omega_{\xi'}(\phi_\beta, \psi_\beta, \chi_\beta, \lambda_\beta)\right]\\ +(\psi'\tau_\alpha+\psi_\alpha)&\left[\tau_\beta\frac{d}{dt}H_{y'}+\Omega_{\eta'}(\phi_\beta, \psi_\beta, \chi_\beta, \lambda_\beta)\right]\end{aligned} \qquad (22)$$

since the values of $\phi'\tau_\alpha+\phi_\alpha$ and $\psi'\tau_\alpha+\psi_\alpha$ at the end-points are independent of x_1, y_1, x_2, y_2, in every case, as follows from equations (8) and (9). The arguments indicated in the derivatives of Ω are substituted for ξ, η, ζ, μ. Similarly, we obtain

$$\begin{aligned}I_{\beta\alpha}=(\phi'\tau_\beta+\phi_\beta)&\left[\tau_\alpha\frac{d}{dt}H_{x'}+\Omega_{\xi'}(\phi_\alpha, \psi_\alpha, \chi_\alpha, \lambda_\alpha)\right]\\ +(\psi'\tau_\beta+\psi_\beta)&\left[\tau_\alpha\frac{d}{dt}H_{y'}+\Omega_{\eta'}(\phi_\alpha, \psi_\alpha, \chi_\alpha, \lambda_\alpha)\right]\Big|_1^2\end{aligned} \qquad (23)$$

To verify the computations, we prove the equality of these expressions for $I_{\alpha\beta}$ and $I_{\beta\alpha}$. By applying the Euler equations (1), and the relations

$$H_{xx'}\phi'+H_{xy'}\psi'=H_x,\qquad H_{yx'}\phi'+H_{yy'}\psi'=H_y,$$
$$H_{x'x'}\phi'+H_{x'y'}\psi'=0,\qquad H_{y'x'}\phi'+H_{y'y'}\psi'=0,$$
$$G_{x'}\phi'+G_{y'}\psi'=G,$$

all of which arise from the homogeneity conditions, we obtain the following:

$$\left.\begin{aligned} I_{\alpha\beta}&=\phi_\alpha\Omega_{\xi'}(\phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta)+\psi_\alpha\Omega_{\eta'}(\phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta)\\ &\quad+\tau_\alpha[H_x\phi_\beta+H_y\psi_\beta+G\lambda_\beta]+(\phi'\tau_\alpha+\phi_\alpha)\tau_\beta H_x+(\psi'\tau_\alpha+\psi_\alpha)\tau_\beta H_y\Big|_1^2,\\ I_{\beta\alpha}&=\phi_\beta\Omega_{\xi'}(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)+\psi_\beta\Omega_{\eta'}(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)\\ &\quad+\tau_\beta[H_x\phi_\alpha+H_y\psi_\alpha+G\lambda_\alpha]+(\phi'\tau_\beta+\phi_\beta)\tau_\alpha H_x+(\psi'\tau_\beta+\psi_\beta)\tau_\alpha H_y\Big|_1^2. \end{aligned}\right\}\quad(24)$$

The notation $\Omega_{\xi'}$, $\Omega_{\eta'}$ is explained specifically in equations (15), reference to which shows that χ_β, χ_α do not actually occur. By the third equation of (15) together with (16), (17) and (18) we have

$$G\tau_\beta\lambda_\alpha\Big|_1^2=\chi_\beta\Omega_{\zeta'}(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)\Big|_1^2,$$
$$G\tau_\alpha\lambda_\beta\Big|_1^2=\chi_\alpha\Omega_{\zeta'}(\phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta)\Big|_1^2.$$

If we make substitutions accordingly in the above expressions for $I_{\alpha\beta}$ and $I_{\beta\alpha}$, and form their difference, we have

$$\begin{aligned} I_{\alpha\beta}-I_{\beta\alpha}&=\phi_\alpha\Omega_{\xi'}(\phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta)-\phi_\beta\Omega_{\xi'}(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)\\ &\quad+\psi_\alpha\Omega_{\eta'}(\phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta)-\psi_\beta\Omega_{\eta'}(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)\\ &\quad+\chi_\alpha\Omega_{\zeta'}(\phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta)-\chi_\beta\Omega_{\zeta'}(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)\\ &=\Psi\begin{pmatrix}\phi_\alpha,\ \psi_\alpha,\ \chi_\alpha,\ \lambda_\alpha,\ \phi'_\alpha,\ \psi'_\alpha,\ \chi'_\alpha,\ \lambda'_\alpha\\ \phi_\beta,\ \psi_\beta,\ \chi_\beta,\ \lambda_\beta,\ \phi'_\beta,\ \psi'_\beta,\ \chi'_\beta,\ \lambda'_\beta\end{pmatrix}\Bigg|_1^2. \end{aligned}$$

But the two sets of arguments in Ψ satisfy equations (15), and hence the function Ψ in the last equation is independent of t. Its values at $t=\tau_1$ and $t=\tau_2$ are the same, and hence it follows that $I_{\alpha\beta}=I_{\beta\alpha}$.

The desired partial derivatives of the extremal integral may now be computed very easily. In the expression (20) for I_α let α take successively the values x_1, y_1, x_2, y_2. Making use of relations (8) and (9) we obtain

$$I_{x_1}=-H_{x'}\Big|^1,\quad I_{y_1}=-H_{y'}\Big|^1,\quad I_{x_2}=+H_{x'}\Big|^2,\quad I_{y_2}=+H_{y'}\Big|^2.\qquad(27)$$

Of the second derivatives ten are required, viz., $I_{x_1x_1}$, $I_{x_1y_1}$, $I_{y_1y_1}$, $I_{x_1y_2}$, $I_{y_1x_2}$, $I_{y_1y_2}$, $I_{x_1x_2}$, $I_{x_2x_2}$, $I_{x_2y_2}$, $I_{y_2y_2}$. With the exception of the cases in which the two subscripts are the same, the derivatives occur in two different forms, whose values, however, are of course the same. The following results are obtained most readily from (22) and (24). We employ relations (8) and (9) and the Euler equations (1).

The former of these two equalities may be written in the form

$$[F_{x'}(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') - H_{x'}]\tilde{x}_u + [F_{y'}(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') - H_{y'}]\tilde{y}_u |^1 = 0,$$

or

$$A\tilde{x}_u + B\tilde{y}_u |^1 = 0,$$

where A and B are the functions defined by (35). Then there exists a function m_1 such that at the point P_1 the following relations hold:

$$A|^1 = \frac{-m_1\tilde{y}_u}{(\sqrt{\tilde{x}_u^2+\tilde{y}_u^2})^3}, \quad B|^1 = \frac{m_1\tilde{x}_u}{(\sqrt{x_u^2+\tilde{y}_u^2})^3}. \tag{36}$$

By a similar argument we have at the point P_2 the relations

$$A|^2 = \frac{-m_2\tilde{y}_v}{(\sqrt{\tilde{x}_v^2+\tilde{y}_v^2})^3}, \quad B|^2 = \frac{m_2\tilde{x}_v}{(\sqrt{\tilde{x}_v^2+\tilde{y}_v^2})^3}. \tag{37}$$

By substitution from (36) and (37) it follows that J_{uu}, J_{uv}, J_{vv} have the forms

$$J_{uu}(u, v) = \frac{m_1}{r_1} + R_1, \quad J_{uv}(u, v) = S_1 = -S_2, \quad J_{vv}(u, v) = -\frac{m_2}{r_2} - R_2,$$

where

$$m_1 = (A^2+B^2)\,|^1\sqrt{\tilde{x}_u^2+\tilde{y}_u^2}, \qquad m_2 = (A^2+B^2)\,|^2\sqrt{\tilde{x}_v^2+\tilde{y}_v^2}, \tag{38}$$

$$\left.\begin{aligned} R_1 &= F_x(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\tilde{x}_u + F_y(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\tilde{y}_u \\ &\qquad -\tilde{x}_u\Omega_{\xi'_1} - \tilde{y}_u\Omega_{\eta'_1} - H_x t_u\tilde{x}_u - H_y t_u\tilde{y}_u |^1, \\ R_2 &= F_x(\tilde{x}, \tilde{y}, x', \tilde{y}')\tilde{x}_v + F_y(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\tilde{y}_v \\ &\qquad -\tilde{x}_v\Omega_{\xi'_2} - \tilde{y}_v\Omega_{\eta'_2} - H_x t_v\tilde{x}_v - H_y t_v\tilde{y}_v |^2, \end{aligned}\right\} \tag{39}$$

$$\left.\begin{aligned} S_1 &= -x_u\Omega_{\xi'_2} - \tilde{y}_u\Omega_{\eta'_2} - H_x t_v\tilde{x}_u - H_y t_v\tilde{y}_u |^1, \\ S_2 &= -\tilde{x}_v\Omega_{\xi'_1} - \tilde{y}_v\Omega_{\eta'_1} - H_x t_u\tilde{x}_v - H_y t_u\tilde{y}_v |^2, \end{aligned}\right\} \tag{40}$$

and r_1 and r_2 are the radii of curvature of L at the points P_1 and P_2, respectively.

If arc E_{12} is to give to the function $J(u, v)$ a maximum value, it is necessary not only that all preceding conditions be satisfied, but also that the second partial derivatives of $J(u, v)$ satisfy the following conditions:

$$J_{uu}(u, v) \leqq 0, \quad J_{vv}(u, v) \leqq 0, \quad J_{uu}(u, v) J_{vv}(u, v) - J_{uv}^2(u, v) \geqq 0.$$

Thus we have a new condition for the problem, analogous to the Jacobi condition in other problems in the Calculus of Variations. We may summarize our results as follows:

Assumed that the Euler, Weierstrass, Legendre and Jacobi conditions are satisfied for the corresponding problem with fixed end-points—the Legendre and Jacobi in the stronger form—and that the arc E_{12} is cut transversally by

the fixed curve L, then a further necessary condition that arc E shall furnish a maximum for the problem at hand is that

$$\left.\begin{array}{ll}\text{(a)} & \dfrac{m_1}{r_1}+R_1 \leqq 0, \quad -\dfrac{m_2}{r_2}-R_2 \leqq 0, \\ \text{(b)} & -\dfrac{m_1m_2}{r_1r_2}-\dfrac{m_1R_2}{r_1}-\dfrac{m_2R_1}{r_2}-R_1R_2+S_1S_2 \geqq 0,\end{array}\right\} \qquad (41)$$

where the notation is explained by equations (38), (39) and (40).

§ 4. *Sufficient Conditions.*

In the determination of conditions which are sufficient, direct application is made of a theorem proved by Hahn.* This theorem holds for the general Lagrangian problem, where certain isoperimetric conditions are to be fulfilled while the end-points of the comparison arcs are to satisfy any number of preassigned conditions. In so far as it relates to the problem at hand, this theorem may be formulated as follows:

Let E_{12} be an extremal arc

$$x=\phi(t), \quad y=\psi(t), \quad t_1 \leqq t \leqq t_2,$$

satisfying the Euler, Weierstrass, Legendre and Jacobi conditions,† the last three in the stronger forms. Then there exist weak neighborhoods $(E_{12})'_\rho$, $(E_{12})'_\sigma$, $\sigma \leqq \rho$, such that every extremal arc E_{34} in $(E_{12})'_\sigma$ furnishes a maximum $I(E_{34})$ for the integral

$$I=\int_{\tau_1}^{\tau_2} F(x, y, x', y')\,dt$$

with respect to all arcs V_{34} in $(E_{12})'_\rho$ such that $K(V_{34})=K(E_{34})$.

Let E_{34} be an extremal arc joining a point $P_3(\varkappa=\varkappa_3)$ to a point $P_4(\varkappa=\varkappa_4)$ on L and having $K(E_{34})=K(E_{12})$. The extremals E_{34} form a two-parameter family with parameters $\varkappa_3$, $\varkappa_4$, containing E_{12} for $\varkappa_3=\varkappa_1$, $\varkappa_4=\varkappa_2$. If the conditions given in the theorem of § 3 are changed by the substitution, in the place of (41), of the conditions

$$\left.\begin{array}{ll}\text{(a)} & \dfrac{m_1}{r_1}-R_1<0, \quad -\dfrac{m_1m_2}{r_1r_2}-R_2<0, \\ \text{(b)} & \dfrac{m_1m_2}{r_1r_2}-\dfrac{m_1R_2}{r_1}-\dfrac{m_2R_1}{r_2}-R_1R_2+S_1S_2>0,\end{array}\right\} \qquad (41')$$

* See Hahn, "Ueber Variations Probleme mit variablen Endpunkten," *Monatshefte für Mathematik und Physik*, Vol. XXII (1911), p. 127.

† These are conditions I, II', III', IV' of Bolza, *loc. cit.*, p. 514 with the proper changes for the determination of a maximum instead of a minimum.

we have a set of conditions which are sufficient to insure that E_{12} furnishes a maximum among the curves E_{34}; *i. e.*, that

$$J(E_{12}) > J(E_{34}), \quad E_{34} \not\equiv E_{12}.$$

Now if $\tau \leqq \sigma$ is sufficiently small, every arc V_{34} with $K(V_{34}) = K(E_{12})$ in $(E_{12})'_\tau$ determines an extremal E_{34} in $(E_{12})'_\sigma$ of the Hahn theorem with $K(E_{34}) = K(V_{34}) = K(E_{12})$, and so near E_{12} that

$$J(E_{34}) < J(E_{12}),$$

according to the preceding paragraph. If then E_{12} satisfies in addition to the above conditions the Weierstrass condition in the stronger form, the hypothesis of the Hahn theorem is satisfied and consequently we have

$$J(V_{34}) < J(E_{34}).$$

Hence it follows that

$$J(V_{34}) < J(E_{12})$$

for all variation arcs V_{34} with $K(V_{34}) = K(E_{12})$ in $(E_{12})'_\tau$. We have thus proved the theorem:

If E_{12} satisfies the Euler condition, the transversality condition, and (41′), *together with the stronger forms of the Weierstrass, Legendre and Jacobi conditions, then there exists a τ such that E_{12} furnishes a maximum $J(E_{12})$ for the function*

$$J = \int_{L_{31}} F(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\, dx + \int_{E_{12}} F(x, y, x', y') \cdot dt$$

with respect to all arcs V_{34} in $(E_{12})'_\tau$ with $K(V_{34}) = K(E_{12})$.

§5. *Geometric Interpretations.*

The conditions found in Section 4 may be interpreted geometrically by the use of a set of oblique axes in the plane. The coordinates of points in the plane referred to this set of axes will be considered as possible values of radii of curvature of the curve L at the points P_1 and P_2. Critical points are those at which the equalities

$$\left.\begin{aligned} &\text{(a)}\quad r_1 + \frac{m_1}{R_1} = 0, \quad r_2 + \frac{m_2}{R_2} = 0, \\ &\text{(b)}\quad r_1 r_2 + \frac{m_2 R_1 r_1}{R_1 R_2 + S_1^2} + \frac{m_1 R_2 r_2}{R_1 R_2 + S_1^2} + \frac{m_1 m_2}{R_1 R_2 + S_1^2} = 0, \end{aligned}\right\} \tag{42}$$

are satisfied, special discussion being required when either r_1 or r_2 vanishes. In any particular case the signs of the various functions are of course

determined. We limit this discussion of the general problem to the case for which
$$m_1>0,\quad R_1<0,\quad m_2<0,\quad R_2>0,\quad c=R_1R_2+S_1^2<0,$$
and will consider that in the given situation, only r_1 and r_2 may vary.

We suppose that in the given situation, the extremal arc E_{12} (see Fig. 3) is cut transversally by the fixed curve L at P_1 and P_2. Through these two points draw lines $X_1'X_2$ and $X_2'X_2$ parallel to r_1 and r_2 respectively, and intersecting* at O. We take these lines as a set of oblique axes, OX_1 and OX_2 being the positive directions. The first two equalities of (42) determine two straight lines $M'M$ and $N'N$ parallel to the axes. The last equality of (42) is

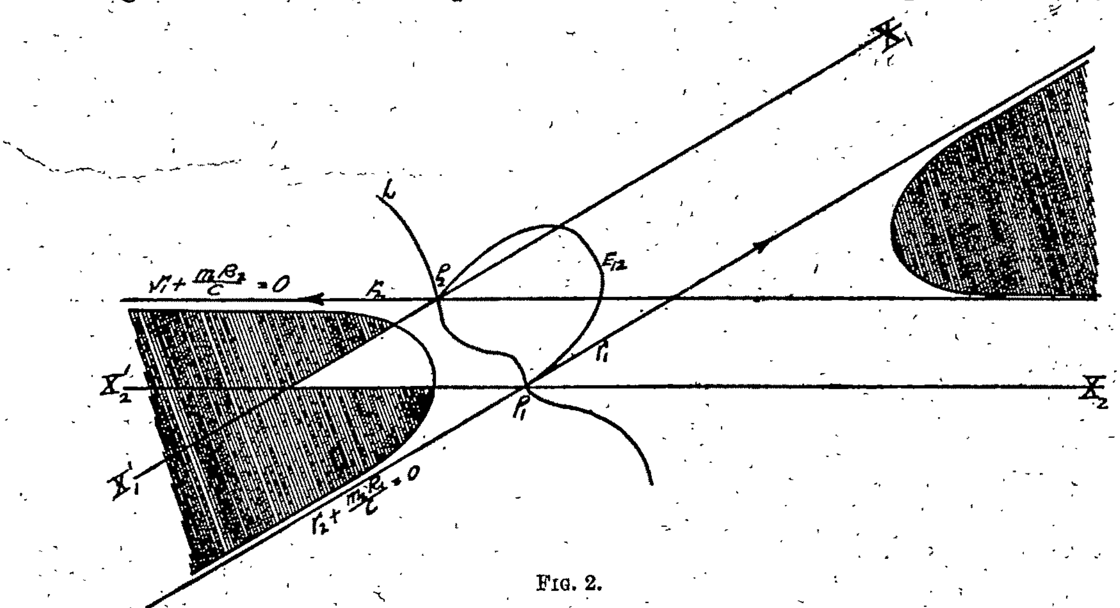

FIG. 2.

an hyperbola lying in the first and third quadrants and asymptotic to the two lines
$$r_1+\frac{m_1R_2}{c}=0,\quad r_2+\frac{m_2R_1}{c}=0.$$

Reference to condition (41) then shows that

In order that two quantities ρ_1, ρ_2, be suitable values for r_1 and r_2 for the existence of a maximum, it is necessary and sufficient that the point (ρ_1, ρ_2) lie in the portion of the plane shaded in Fig. 2.

§ 6. *Application to the Problem of Dido.*

Let L be a given fixed curve. We wish to determine an arc E_{12} with end-points P_1 and P_2 on L and with a given length k, such that the area enclosed

* In case these lines do not intersect, some other set of lines may be used; for instance, line r_1 and the line perpendicular to it through P.

by the arc P_2P_1 of L and the arc P_1P_2 of E shall be a maximum, P_1 and P_2 to be distinct points.

The functions F and G for this problem are

$$F(x, y, x', y') = \tfrac{1}{2}(xy' - yx'), \qquad G(x, y, x', y') = \sqrt{x'^2 + y'^2}.$$

Suppose that the equations of the fixed curve L are

$$x = \tilde{x}(\varkappa), \quad y = \tilde{y}(\varkappa), \tag{L}$$

and denote by u and v the values of $\varkappa$ in the neighborhoods of the values of $\varkappa_1$ and $\varkappa_2$, which latter values define P_1 and P_2, respectively. Let the equation of E_{12} be

$$x = \phi(t), \quad y = \psi(t), \quad t_1 \leqq t \leqq t_2. \tag{E}$$

We have then to maximize the function

$$J = \tfrac{1}{2}\int_v^u (\tilde{x}\tilde{y}' - \tilde{y}\tilde{x}')\, d\varkappa + \tfrac{1}{2}\int_{t_1}^{t_2} (\phi\psi' - \psi\phi')\, dt,$$

while the integral K:

$$K = \int_{t_1}^{t_2} \sqrt{\phi'^2 + \psi'^2}\, dt$$

is to have a preassigned value k.

Applying the results of §§ 1, 3, 4, we obtain the following conditions for the problem:

I. E_{12} is the arc of a circle*

$$x = \alpha - \lambda \cos t, \quad y = \beta - \lambda \sin t, \quad t_1 \leqq t \leqq t_2.$$

II. $$E(x, y, p, q, x', y'; \lambda) = \lambda \frac{(px' - qy')^2}{\sqrt{p^2 + q^2}\{\sqrt{p^2 + q^2}\sqrt{x'^2 + y'^2} + px' + qy'\}} \leqq 0,$$

i. e., $\lambda \leqq 0$ for (x, y, p, q) on E_{12} for every $(x', y') \neq (0, 0)$. Since by condition I, λ is seen to be the radius of the circular arc E_{12}, the value zero is necessarily excluded, and we have the stronger condition $\lambda < 0$.

III. $H_1 = \dfrac{\lambda}{(\sqrt{x'^2 + y'^2})^3} \leqq 0$ along E_{12}. This condition follows directly from condition II, and in fact in the stronger form.

IV. E_{12} contains no conjugate point to P_1 or P_2; that is, since P_1 and P_2 are distinct, $t_2 < t_1 + 2\pi$.

Now from I,

$$\sqrt{\phi'^2 + \psi'^2} = -\lambda.$$

Since E and L intersect at P_1 we have

$$\phi'|^1 = \lambda \sin t_1, \quad \psi'|^1 = -\lambda \cos t_1.$$

* As to the determination of conditions I–V, cf. also Bolza, *loc. cit.*, pp. 465, 483.

Accordingly we have for the transversality condition,

$$\tilde{x}'\sin t_1-\tilde{y}'\cos t_1=0$$

and a similar result at P_2. We therefore have the condition

V. E_{12} cuts the curve L orthogonally at both P_1 and P_2.

If now the length of arc is chosen as the parameter x, we have from condition V the important relations:

$$\tilde{x}_u(u)=\cos t_1,\quad \tilde{y}_u(u)=\sin t_1,$$
$$\tilde{x}_v(v)=-\cos t_2,\quad \tilde{y}_v(v)=-\sin t_2.$$

Employing these relations and the values of $\alpha_u, \beta_u, \alpha_v, \beta_v$ from the Euler equations, we obtain from the last three expressions of (34) the following expressions for the second partial derivatives:

$$J_{uu}(u,v)=-\frac{\lambda}{r_1}-\frac{D_2}{D_1},\quad J_{uv}(u,v)=\frac{D_3}{D_1},\quad J_{vv}(u,v)=-\frac{\lambda}{r_2}-\frac{D_2}{D_1},$$

where

$$D_1=2-2\cos(t_2-t_1)-(t_2-t_1)\sin(t_2-t_1),$$
$$D_2=-\sin(t_2-t_1)+(t_2-t_1)\cos(t_2-t_1),$$
$$D_3=(t_2-t_1)-\sin(t_2-t_1),$$
$$r_1=\frac{1}{\tilde{x}_u\tilde{y}_{uu}-\tilde{y}_u\tilde{x}_{uu}|^1},\quad r_2=\frac{1}{\tilde{x}_v\tilde{y}_{vv}-\tilde{y}_v\tilde{x}_{vv}|^2},$$

r_1 and r_2 being the radii of curvature of the curve L at P_1 and P_2, respectively. Conditions (41) are therefore*

$$-\frac{\lambda}{r_1}-\frac{D_2}{D_1}\leqq 0,\quad -\frac{\lambda}{r_2}-\frac{D_2}{D_1}\leqq 0,\quad \left[-\frac{\lambda}{r_1}-\frac{D_2}{D_1}\right]\cdot\left[-\frac{\lambda}{r_2}-\frac{D_2}{D_1}\right]-\frac{D_3^2}{D_1^2}\geqq 0.$$

It is evident that if the first and third of these conditions are satisfied, the second must also hold.

We may state the result as follows:

In order that E_{12} and L_{21} enclose a maximum area in the Problem of Dido stated above, it is necessary that E_{12} be a circle-arc

$$x=\alpha-\lambda\cos t,\quad y=\beta-\lambda\sin t,\quad t_1\leqq t\leqq t_2,$$

with $\lambda<0$ and $t_2<t_1+2\pi$, cutting L orthogonally at P_1 and P_2, and that the conditions

$$-\frac{\lambda}{r_1}-\frac{D_2}{D_1}\leqq 0,\quad \left(-\frac{\lambda}{r_1}-\frac{D_2}{D_1}\right)\left(-\frac{\lambda}{r_2}-\frac{D_2}{D_1}\right)-\frac{D_3^2}{D_1^2}\geqq 0$$

be fulfilled.

Applying the results of § 4, we have immediately the following:

* With the first two of these conditions cf. Bolza, *loc. cit.*, p. 538, ex. 29.

If E_{12} is a circle-arc

$$x=\alpha-\lambda\cos t,\quad y=\beta-\lambda\sin t,\quad t_1\leq t\leq t_2,$$

with $\lambda<0$ and $t_2<t_1+2\pi$, cutting L orthogonally at P_1 and P_2, and if in addition the conditions

$$-\frac{\lambda}{r_1}-\frac{D_2}{D_1}<0,\quad \left(-\frac{\lambda}{r_1}-\frac{D_2}{D_1}\right)\left(-\frac{\lambda}{r_2}-\frac{D_2}{D_1}\right)-\frac{D_3^2}{D_1^2}>0$$

be fulfilled, then there exists a τ such that the area enclosed by arcs L_{21} and E_{12} is the greatest among the areas enclosed by arcs L_{43} and V_{34}, for all arcs V_{34} in $(E_{12})'_\tau$ and of the same length as E_{12}.

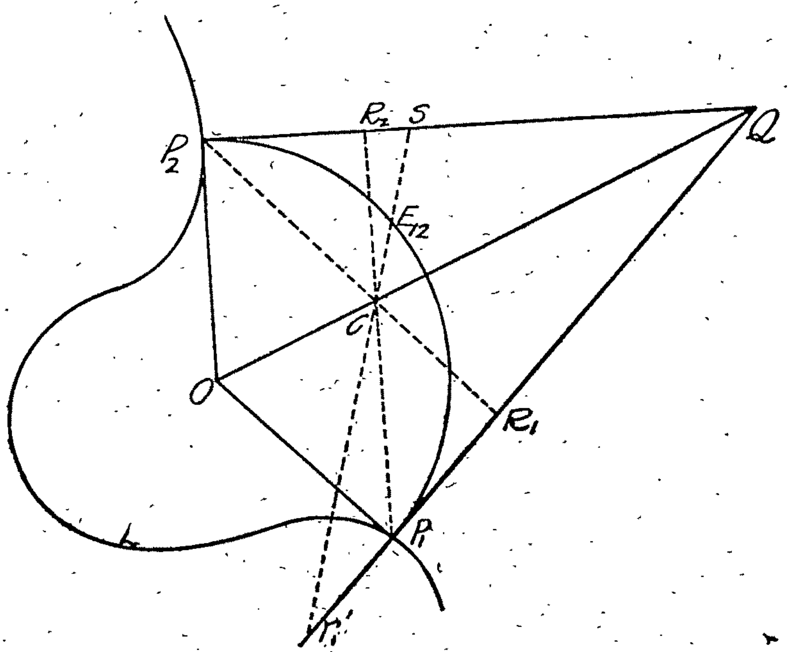

FIG. 3.

The geometric interpretation of § 5 is of course applicable to the Problem of Dido, but for this particular case, the following is a more direct interpretation. Suppose L and E_{12} are as in Fig. 3. P_1Q and P_2Q are tangents to E_{12}. Also

$$\angle P_1OQ=QOP_2=\omega=\frac{t_2-t_1}{2}.$$

If we measure r_1 from P_1 along P_1Q, and r_2 from P_2 along P_2Q, positive directions being P_1 to Q and P_2 to Q, then the relation

$$\left(\frac{\lambda}{r_1}+\frac{D_2}{D_1}\right)\left(\frac{\lambda}{r_2}+\frac{D_2}{D_1}\right)-\frac{D_3^2}{D_1^2}=0 \tag{43}$$

relates projectively the points of the line P_1Q to those of P_2Q. This pro-

jectivity is in fact a perspectivity, the center C of the perspectivity being a point on OQ at the distance $\frac{|\lambda|\sin\omega}{\omega}$ from O.

Let R_1 and R_2 be the points on P_1Q and P_2Q determined by P_2C and P_1C, respectively. Then the condition

$$-\frac{\lambda}{r_1}-\frac{D_2}{D_1}<0$$

means that r_1 is not on the segment P_1R_1. Similarly, the condition

$$-\frac{\lambda}{r_2}-\frac{D_2}{D_1}<0$$

being fulfilled means that r_2 does not lie on P_2R_2.

Suppose now that a particular value r_1' of r_1 be given, determining the point r_1' in the figure. Draw $r_1'C$ cutting P_2Q at S. Then the condition

$$\left(\frac{\lambda}{r_1}+\frac{D_2}{D_1}\right)\left(\frac{\lambda}{r_2}+\frac{D_2}{D_1}\right)-\frac{D_3^2}{D_1^2}>0$$

means that r_2' may not lie on the segment P_2S.

It remains to prove the statement made above, that (43) relates perspectively the points of P_1Q to those of P_2Q. In the figure take OX as the positive x-axis, O being the origin, of a set of perpendicular axes. Then P_1 is the point $(|\lambda|\cos\omega, -|\lambda|\sin\omega)$, while P_2 is $(|\lambda|\cos\omega, |\lambda|\sin\omega)$ and C is $\left(\frac{|\lambda|\sin\omega}{\omega}, 0\right)$. The point on P_1Q at the distance r_1 from P_1 is

$$(|\lambda|\cos\omega+r_1\sin\omega,\ -|\lambda|\sin\omega+r_1\cos\omega),$$

and the point on P_2Q at the distance r_2 from P_2 is

$$(|\lambda|\cos\omega+r_2\sin\omega,\ |\lambda|\sin\omega-r_2\cos\omega).$$

The condition that these two latter points be collinear with C is

$$\begin{vmatrix} |\lambda|\cos\omega+r_1\sin\omega & |\lambda|\cos\omega+r_2\sin\omega & \frac{|\lambda|\sin\omega}{\omega} \\ -|\lambda|\sin\omega-r\cos\omega & |\lambda|\sin\omega-r_2\cos\omega & 0 \\ 1 & 1 & 1 \end{vmatrix}=0.$$

After expansion and reduction this condition is found to be equivalent to relation (43).

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Asymptotic Satellites near the Straight-Line Equilibrium Points in the Problem of Three Bodies.

BY DANIEL BUCHANAN.

§ 1. OSCILLATING SATELLITES.

It was shown by Lagrange in a prize memoir * in 1772 that if two finite spheres revolve in circles about their common centre of mass, then there are three points on the line joining their centres at which an infinitesimal body would remain if it were given an initial projection so as to be instantaneously fixed with respect to the moving bodies. These points are called the *straight-line equilibrium points* of the problem of three bodies. Starting from minus infinity the order of the equilibrium points and the finite bodies is an equilibrium point, a finite body, a second equilibrium point, the other finite body, the third equilibrium point.

If the infinitesimal body is given an initial displacement from a point of equilibrium and initial conditions are so chosen that it moves in a closed orbit relatively to the moving system, it is then called an *oscillating satellite.*

The problem of the oscillating satellite has been discussed extensively. In the papers cited below,† the differential equations are limited to their linear terms and the orbits restricted to the plane of motion of the finite bodies.

A rigorous demonstration for the existence of periodic orbits for the oscillating satellite and a practical method for constructing them are given by Moulton in Chapter V of his "Periodic Orbits." ‡ In this memoir the differential equations are unrestricted in the number of terms which may be taken,

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* Lagrange, "Collected Works," Vol. VI, pp. 229–324.

† The following references to the literature of the oscillating satellite are taken from Chapter V of Moulton's "Periodic Orbits":

Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. I (1892), p. 159.

Burrau, *Astronomische Nachrichten*, Nos. 3230, 3251 (1894).

Perchot and Mascart, *Bulletin Astronomique*, Vol. XII (1895), p. 329. Moulton adds, "apparently their work is vitiated by an error in establishing the existence of the solutions, and their construction fails where they stopped."

Sir George H. Darwin, *Acta Mathematica*, Vol. XXI (1897), p. 99.

Plummer, *Monthly Notices, Royal Astronomical Society*, Vol. LXIII (1903), p. 436, and Vol. LXIV (1903), p. 98.

‡ This memoir will be cited as the "Oscillating Satellite." Another method is given in Chapter VI of the "Periodic Orbits," but we are concerned with the former method only.

and the orbits are not limited to two dimensions as in the previous literature. Three classes of orbits are shown to exist and they are designated according to their periods as Class A, Class B, and Class C. The orbits of Class A and Class C are of three dimensions, while those of Class B are of two dimensions. The orbits of Class C are shown to exist under special conditions which might never be realized in the problem. Their period is a multiple of the periods of Class A and Class B when these latter periods are commensurable. Practical constructions are made for the orbits of Class A and of Class B, but, owing to the complexity of the problem, no attempt has been made to determine whether orbits of Class C exist which are distinct from those of Class A and Class B.

§ 2. ASYMPTOTIC SATELLITES.

The object of this paper is to obtain solutions of the differential equations of motion of the infinitesimal body which will approach the periodic solutions of Class A and Class B as the time approaches plus infinity or minus infinity. When the infinitesimal body moves in an orbit defined by such solutions, it will be called an *asymptotic satellite.*

The question of the existence of solutions which are asymptotic to the orbits of Class C is not considered in this paper, owing to the fact that it has not been determined in the "Oscillating Satellite" that orbits of Class C exist which are distinct from those of Class A and Class B.

The orbits which are asymptotic to the equilibrium points themselves have been determined by Warren.* These orbits are of two dimensions and lie in the plane of motion of the finite bodies.

The form of the asymptotic solutions is that adopted by Poincaré,† each term being of the type $e^{\lambda t}P(t)$, where λ is a constant having its real part different from zero, and $P(t)$ is a constant or periodic function of t. It has been shown by Poincaré that solutions of this type will converge for all values of t, provided that certain divisors which appear in the construction of such solutions are different from zero.‡ If these divisors vanish, terms of the form $te^{\lambda t}P(t)$ will arise. If, therefore, the construction can be made so that no terms occur in t explicitly, the divisors previously mentioned are different from zero and the solutions will converge for all values of t. Hence it is sufficient

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* Warren, "A Class of Asymptotic Orbits in the Problem of Three Bodies," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVIII, No. 3 (1916), pp. 221–248.

† Poincaré, "Mécanique Celéste," Vol. I, p. 340.

‡ Poincaré, *loc. cit.* p. 341.

to consider only the formal construction of the asymptotic solutions, and if solutions can be constructed so as to contain no terms in t explicitly, their convergence is assured by Poincaré's theorem.

§ 3. The Differential Equations of Motion.

Let the motion of the infinitesimal body be referred to a set of rotating rectangular coordinates $\xi\eta\zeta$ of which the origin is at the centre of mass of the finite bodies, the ξ-axis is the line joining the finite bodies, and the $\xi\eta$-plane is the plane of their motion. The ξ- and η-axes rotate about the ζ-axis in the direction of the motion of the finite bodies and with the same angular velocity. The units of length, mass, and time will be taken so that the distance between the finite bodies, the sum of their masses, and the Gaussian constant respectively shall each be unity. With the units thus chosen, the mean angular motion of the system is likewise unity. Let the masses of the finite bodies be denoted by $1-\mu$ and μ, $0<\mu\leqq\frac{1}{2}$. On denoting the coordinates of the infinitesimal body by ξ, η, ζ, and differentiation with respect to t by primes, the differential equations of motion are*

$$\left.\begin{aligned}
&\xi''-2\eta'=\frac{\partial U}{\partial\xi},\quad \eta''+2\xi'=\frac{\partial U}{\partial\eta},\quad \zeta''=\frac{\partial U}{\partial\zeta},\\
&2U=\xi^2+\eta^2+\frac{2(1-\mu)}{r_1}+\frac{2\mu}{r_2}\\
&\quad=(1-\mu)\left(r_1^2+\frac{2}{r_1}\right)+\mu\left(r_2^2+\frac{2}{r_2}\right)-\zeta^2-\mu(1-\mu),\\
&r_1=\sqrt{(\xi+\mu)^2+\eta^2+\zeta^2},\quad r_2=\sqrt{(\xi-1+\mu)^2+\eta^2+\zeta^2}.
\end{aligned}\right\} \tag{1}$$

The points of equilibrium are the solutions of the equations †

$$\frac{\partial U}{\partial\xi}=\frac{\partial U}{\partial\eta}=\frac{\partial U}{\partial\zeta}=0.$$

There are two sets of points which satisfy these equations. One set consists of the two points which are at the vertices of the two equilateral triangles on the opposite sides of the line joining the finite bodies. The orbits which are asymptotic to these points and also to the periodic oscillations near these points are discussed in another paper.‡ The other set of points consists of

* Moulton, "Introduction to Celestial Mechanics" (1914), p. 279.

† Moulton, "Introduction to Celestial Mechanics," p. 290; Charlier, "Die Mechanik des Himmels," Vol. II, pp. 102–111.

‡ This paper is now under consideration for publication.

three points which lie on the straight line joining the finite bodies. Let the coördinates of these points be denoted by ξ_0, 0, 0, where the particular value of ξ_0 depends upon the equilibrium point in question.* The points themselves will be denoted by (*a*), (*b*), and (*c*), where (*a*) lies between $+\infty$ and the finite mass μ, (*b*) between μ and $1-\mu$, and (*c*) between $1-\mu$ and $-\infty$.

If the infinitesimal body is given a small displacement from an equilibrium point and a small velocity with respect to the finite masses such that

$$\left.\begin{array}{lll}\xi=\xi_0+\bar{x}, & \eta=0+\bar{y}, & \zeta=0+\bar{z},\\ \xi'=0+\bar{x}', & \eta'=0+\bar{y}', & \zeta=0+\bar{z}',\end{array}\right\} \tag{2}$$

then the differential equations (1) become†

$$\left.\begin{aligned}
\bar{x}''-2\bar{y}' &= \frac{\partial U}{\partial \bar{x}} = +X(\bar{x}, \bar{y}^2, \bar{z}^2),\\
\bar{y}''+2\bar{x}' &= \frac{\partial U}{\partial \bar{y}} = \bar{y}Y(\bar{x}, \bar{y}^2, \bar{z}^2),\\
\bar{z}'' &= \frac{\partial U}{\partial \bar{z}} = \bar{z}Z(\bar{x}, \bar{y}^2, \bar{z}^2),\\
U &= \tfrac{1}{2}(1-\mu)\left(r_1^2+\frac{2}{r_1}\right)+\tfrac{1}{2}\mu\left(r_2^2+\frac{2}{r_2}\right)-\tfrac{1}{2}\bar{z}^2-\tfrac{1}{2}\mu(1-\mu),\\
r_1 &= \sqrt{(\xi_0+\bar{x}+\mu)^2+\bar{y}^2+\bar{z}^2}, \quad r_2=\sqrt{(\xi_0-1+\bar{x}+\mu)^2+\bar{y}^2+\bar{z}^2},
\end{aligned}\right\} \tag{3}$$

where X, Y, and Z are power series in $\bar{x}$, $\bar{y}^2$, $\bar{z}^2$. These series converge within certain regions about the equilibrium points.‡

§ 4. THE PERIODIC ORBITS.

In showing the existence of periodic solutions of the differential equations of the "Oscillating Satellite" which correspond to equations (3), and later in making the construction of these solutions, the transformations

$$\bar{x}=\varepsilon x, \quad \bar{y}=\varepsilon y, \quad \bar{z}=\varepsilon z, \quad t-t_0=(1+\delta)\tau \tag{4}$$

are made, where ε is an arbitrary parameter and δ is determined as a function of ε so that the solutions in x, y, and z shall be periodic with the periodic 2π in τ. On denoting differentiation with respect to τ by a dot over the variables, the differential equations (3) become as a consequence of (4)§

$$\left.\begin{aligned}
\ddot{x}-2(1+\delta)\dot{y} &= (1+\delta)^2[X_1+X_2\varepsilon+\ldots+X_k\varepsilon^{k-1}+\ldots],\\
\ddot{y}+2(1+\delta)\dot{x} &= (1+\delta)^2[Y_1+Y_2\varepsilon+\ldots+Y_k\varepsilon^{k-1}+\ldots],\\
\ddot{z} &= (1+\delta)^2[Z_1+Z_2\varepsilon+\ldots+Z_k\varepsilon^{k-1}+\ldots],
\end{aligned}\right\} \tag{5}$$

* "Oscillating Satellite," equations (4).

† "Oscillating Satellite," equations (6).

‡ "Oscillating Satellite," § 77.

§ "Oscillating Satellite," equations (11).

where X_k, Y_k, and Z_k are homogeneous polynomials of degree k in x, y, and z. From (3) it is obvious that the X_k are even in y and z, Y_k odd in y and even in z, and Z_k even in y and odd in z. The explicit values of these terms up to $k=3$ are*

$$\begin{aligned}
&X_1=(1+2A)x, && X_2=\tfrac{3}{2}B(-2x^2+y^2+z^2), && X_3=2C(2x^3-3xy^2-3xz^2),\\
&Y_1=(1-A)y, && Y_2=3Bxy, && Y_3=\tfrac{3}{2}C(-4x^2y+y^3+yz^2),\\
&Z_1=-Az, && Z_2=3Bxz, && Z_3=\tfrac{3}{2}C(-4x^2z+y^2z+z^3),
\end{aligned}$$

$$A=\frac{1-\mu}{r_1^{(0)3}}+\frac{\mu}{r_2^{(0)3}},\quad r_1^{(0)}=+\sqrt{(\xi_0+\mu)^2},$$

$$B=\pm\frac{1-\mu}{r_1^{(0)4}}\pm\frac{\mu}{r_2^{(0)4}},\quad r_2^{(0)}=+\sqrt{(\xi_0-1+\mu)^2},$$

$$C=\frac{1-\mu}{r_1^{(0)5}}+\frac{\mu}{r_2^{(0)5}}.$$

The upper, middle, or lower signs are to be taken in B according as the equilibrium point is (a), (b), or (c) respectively.

If periodic solutions of (5) exist, their periods are determined from periods of the solutions of the linear terms of (5). The solutions of the linear terms are†

$$\left.\begin{aligned}
&x=K_1e^{i\sigma\tau}+K_2e^{-i\sigma\tau}+K_3e^{\rho\tau}+K_4e^{-\rho\tau}, \quad i=\sqrt{-1},\\
&y=in(K_1e^{i\sigma\tau}-K_2e^{-i\sigma\tau})+m(K_3e^{\rho\tau}-K_4e^{-\rho\tau}),\\
&z=K_5\cos\sqrt{A}\tau+K_6\sin\sqrt{A}\tau,\quad n=\frac{\sigma^2+1+2A}{2\sigma},\quad m=\frac{\rho^2-1-2A}{2\rho},
\end{aligned}\right\}\tag{6}$$

where $K_1, \ldots, K_6$ are the constants of integration, and σ^2 and ρ^2 are the negative and positive roots respectively of the quadratic in λ^2,

$$\lambda^4+(2-A)\lambda^2+(1-A)(1+2A)=0.\tag{7}$$

There are three real periods in these solutions, viz., $2\pi/\sqrt{A}$, $2\pi/\sigma$, and $P=2j\pi/\sqrt{A}=2k\pi/\sigma$, where j and k are positive integers and $\sqrt{A}$ and σ are commensurable.‡ These are the periods of the orbits of Class A, Class B, and Class C respectively.

Orbits of Class A.

The periodic solutions of Class A are§

$$\left.\begin{aligned}
&\bar{x}_1=\varepsilon x_1=0\varepsilon+(a_1+b_1\cos 2\sqrt{A}\tau)\varepsilon^2+0\varepsilon^3+\ldots,\\
&\bar{y}_1=\varepsilon y_1=0\varepsilon+(c_1\sin 2\sqrt{A}\tau)\varepsilon^2+0\varepsilon^3+\ldots,\\
&\bar{z}_1=\varepsilon z_1=\left(\frac{1}{\sqrt{A}}\sin\sqrt{A}\tau\right)\varepsilon+0\varepsilon^2+d_1(3\sin\sqrt{A}\tau-\sin 3\sqrt{A}\tau)\varepsilon^3+\ldots,\\
&\delta=\delta_1=0\varepsilon+\delta_1^{(2)}\varepsilon^2+\ldots+\delta_1^{(2j)}\varepsilon^{2j},
\end{aligned}\right\}\tag{8}$$

* "Oscillating Satellite," equations (53), (15), (35).

† "Oscillating Satellite," equations (16), (27), and (28).

‡ In §83 of the "Oscillating Satellite" it is shown that there are infinitely many values of μ in 0 and $\frac{1}{2}$ such that $\sqrt{A}$ and σ are commensurable.

§ "Oscillating Satellite," equations (71).

where
$$a_1=\frac{-3B}{4A(1+2A)}, \qquad b_1=\frac{3B(1+3A)}{4A(1-7A+18A^2)},$$
$$c_1=\frac{-3B}{\sqrt{A}(1-7A+18A^2)}, \quad d_1=\frac{3}{64A^{5/2}}\left[\frac{3B^2(1+3A)}{1-7A+18A^2}-C\right],$$
$$\delta_1^{(2)}=-\frac{9}{16A^2}\left[\frac{3B^2(1-3A+14A^2)}{(1+2A)(1-7A+18A^2)}-C\right].$$

The expressions for $\bar{x}_1$ and $\bar{y}_1$ are power series in ε^2 with sums of cosines and sines respectively of even multiples of $\sqrt{A}\tau$ in the coefficients. The highest multiple of $\sqrt{A}\tau$ which appears in the coefficient of ε^k in each series is k. The solution for $\bar{z}_1$ is a power series in odd powers of ε having sums of sines of odd multiples of $\sqrt{A}\tau$ in the coefficients. The highest multiple of $\sqrt{A}\tau$ which occurs in the coefficient of ε^k is likewise k. The initial conditions are so chosen in the construction of these solutions that

$$\bar{z}(0)=0, \quad \dot{\bar{z}}(0)=\varepsilon,$$

is, the parameter ε is proportional to the initial projection of the infinitesimal body from the plane of motion. The actual projection is $\varepsilon/(1+\delta_1)$.

Numerical examples of these orbits have been considered in the "Oscillating Satellite" and the following results have been obtained when the ratio of the finite masses is ten to one, or $1-\mu=10/11$, $\mu=1/11$.*

TABLE 1.

Coefficient.	Point (*a*).	Point (*b*).	Point (*c*).
A	2.548	6.510	1.082
σ^2	2.811	6.820	1.144
ρ^2	3.359	11.330	0.226
n	2.657	3.990	2.014
m	−0.747	− 0.397	−3.091
B	6.548	−10.961	−1.136
C	18.283	55.740	1.196
a_1	−0.316	0.090	0.249
b_1	0.151	− 0.036	−0.230
c_1	−0.112	0.018	0.226
d_1	−0.037	− 0.020	−0.002
$\delta_1^{(2)}$	0.184	0.467	0.001

* This was the ratio used by Darwin in the memoir cited at the beginning of this article. The same ratio was adopted by Moulton in the "Oscillating Satellites" so that he might compare his results with those of Darwin.

Orbits of Class B.

The periodic solutions of Class B are*

$$\left.\begin{aligned} \bar{x}_2 &= \varepsilon x_2 = (\cos \sigma\tau)\varepsilon + (a_2 + b_2 \cos \sigma\tau + c_2 \cos 2\sigma\tau)\varepsilon^2 + \ldots, \\ \bar{y}_2 &= \varepsilon y_2 = (-n \sin \sigma\tau)\varepsilon + (-nb_2 \sin \sigma\tau + d_2 \sin 2\sigma\tau)\varepsilon^2 + \ldots, \\ \delta &= \delta_2 = 0\varepsilon + \delta_2^{(2)}\varepsilon^2 + \delta_2^{(3)}\varepsilon^3 + \ldots, \end{aligned}\right\} \tag{9}$$

where a_2, b_2, c_2, d_2, $\delta_2^{(2)}$, ..., are known constants. The algebraic expressions for these constants may be found in equations (101), (103), and (107) of the "Oscillating Satellite." Their particular values for $1-\mu=10/11$, $\mu=1/11$ are to be found in Table 2. The first seven coefficients which appear in Table 1 have been used in determining the values in Table 2.

TABLE 2.

Coefficient.	Point (*a*).	Point (*b*).	Point (*c*).
a_2	−4.078	8.172	0.554
b_2	1.996	− 4.976	−0.038
c_2	2.082	− 3.196	−0.516
nb_2	5.305	−19.855	−0.077
d_2	1.250	− 1.478	−0.276
$\delta_2^{(2)}$	3.955	8.553	−1.407

The expressions for $\bar{x}_2$ and $\bar{y}_2$ in (9) are power series in ε with sums of cosines and sines respectively of $\sigma\tau$ in the coefficients. The highest multiple of $\sigma\tau$ which occurs in the coefficient of ε^k is k. The initial conditions are so chosen for the construction of these solutions that

$$\bar{x}_2(0) = \varepsilon, \quad \bar{y}_2(0) = 0.$$

In this case the parameter ε denotes the initial displacement along the ξ-axis of the infinitesimal body from an equilibrium point.

§ 5. The Equations of Variation.

Suppose now that the initial conditions for the periodic orbits are changed slightly. Then the infinitesimal body will deviate from its periodic orbit, and the amount and character of the deviation will depend upon the initial conditions chosen. The equations for the disturbed orbit, in terms of the variables x, y, and z, will be

$$x = x_j + p_j, \quad y = y_j + q_j, \quad z = z_j + r_j\dagger, \tag{10}$$

* "Oscillating Satellite," equations (97), (105).

† The r_1 and r_3 in this equation and in subsequent equations are different from r_1 and r_3 in equations (1) and (3).

where εx_j, εy_j, and εz_j represent the periodic orbits of Class A or Class B according as $j=1$ or 2 respectively. The additional terms p_j, q_j, and r_j are functions of τ which depend upon the initial disturbance from the periodic orbit. We desire to determine these functions so that they shall have the asymptotic form already referred to in § 2.

On substituting (10) in (5), the differential equations which define p_j, q_j, and r_j are found to be

$$\left.\begin{aligned} \ddot{p}_j-2(1+\delta_j)\dot{q}_j+(1+\delta_j)^2(P_{j1}p_j+P_{j2}q_j+P_{j3}r_j)&=(1+\delta_j)^2P_j,\\ \ddot{q}_j+2(1+\delta_j)\dot{p}_j+(1+\delta_j)^2(Q_{j1}p_j+Q_{j2}q_j+Q_{j3}r_j)&=(1+\delta_j)^2Q_j,\\ \ddot{r}_j+(1+\delta_j)^2(R_{j1}p_j+R_{j2}q_j+R_{j3}r_j)&=(1+\delta_j)^2R_j, \end{aligned}\right\} \tag{11}$$

where

$$\begin{aligned} P_{j1}&=-1-2A+6B\varepsilon x_j-6C\varepsilon^2(2x_j^2-y_j^2-z_j^2)+\ldots,\\ P_{j2}&=Q_{j1}=-3B\varepsilon y_j+12C\varepsilon^2x_jy_j+\ldots,\\ P_{j3}&=R_{j1}=-3B\varepsilon z_j+12C\varepsilon^2x_jz_j+\ldots,\\ Q_{j2}&=-1+A-3B\varepsilon x_j+\tfrac{3}{2}C\varepsilon^2(4x_j^2-3y_j^2-z_j^2)+\ldots,\\ Q_{j3}&=R_{j2}=-3C\varepsilon^2y_jz_j+\ldots,\\ R_{j3}&=A-3B\varepsilon x_j+\tfrac{3}{2}C\varepsilon^2(4x_j^2-y_j^2-3z_j^2)+\ldots,\\ P_j&=\tfrac{3}{2}B\varepsilon(-2p_j^2+q_j^2+r_j^2)+(\,)\varepsilon^2+\ldots,\\ Q_j&=3B\varepsilon p_jq_j+(\,)\varepsilon^2+\ldots,\\ R_j&=3B\varepsilon p_jr_j+(\,)\varepsilon^2+\ldots. \end{aligned}$$

As we propose later to construct numerical solutions which are asymptotic to the particular periodic orbits mentioned above, we insert here the particular values of the various P_{jk}, Q_{jk}, and R_{jk} which are obtained by making use of the numerical values listed in Tables 1 and 2. The following results are found.

Orbits of Class A.

Equilibrium point (*a*).

$$\begin{aligned} P_{11}&=-6.096+\varepsilon^2(8.384-15.686\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,\\ P_{12}&=Q_{21}=\varepsilon^2(2.200\sin 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,\\ P_{13}&=R_{11}=-\varepsilon(12.297\sin\sqrt{A}\tau)+(\,)\varepsilon^3+\ldots,\\ Q_{12}&=1.548+\varepsilon^2(0.833+2.409\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,\\ Q_{13}&=R_{12}=(\,)\varepsilon^3+\ldots\\ R_{13}&=2.548-\varepsilon^2(9.918-13.160\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots. \end{aligned}$$

Equilibrium point (*b*).

$$P_{11}=-14.020+\varepsilon^2(19.777-23.328\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,$$
$$P_{12}=Q_{21}=\varepsilon^2(0.592\sin 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,$$
$$P_{13}=R_{11}=\varepsilon(12.890\sin \sqrt{A}\tau)+(\,)\varepsilon^3+\ldots,$$
$$Q_{12}=5.510+\varepsilon^2(-3.464+5.240\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,$$
$$Q_{13}=R_{12}=(\,)\varepsilon^3+\ldots,$$
$$R_{13}=6.510+\varepsilon^2(-16.312+18.108\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots.$$

Equilibrium point (*c*).

$$P_{11}=-3.164+\varepsilon^2(1.616-1.746\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,$$
$$P_{12}=Q_{21}=\varepsilon^2(0.770\sin 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,$$
$$P_{13}=R_{11}=\varepsilon(3.275\sin \sqrt{A}\tau)+(\,)\varepsilon^3+\ldots,$$
$$Q_{12}=0.082+\varepsilon^2(0.020+0.044\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,$$
$$Q_{13}=R_{12}=(\,)\varepsilon^3+\ldots,$$
$$R_{13}=1.082+\varepsilon^2(0.020+0.044\cos 2\sqrt{A}\tau)+\ldots.$$

Orbits of Class B.

Equilibrium point (*a*).

$$P_{21}=-6.10+\varepsilon(39.30\cos\sigma\tau)+\ldots,\qquad P_{22}=Q_{21}=\varepsilon(52.27\sin\sigma\tau)+\ldots,$$
$$Q_{22}=1.55-\varepsilon(19.64\cos\sigma\tau)+\ldots,\qquad P_{23}=Q_{23}=R_{21}=R_{22}=0,$$
$$R_{23}=2.55-\varepsilon(19.64\cos\sigma\tau)+\ldots.$$

Equilibrium point (*b*).

$$P_{21}=-14.02-\varepsilon(65.77\cos\sigma\tau)+\ldots,\qquad P_{22}=Q_{21}=-\varepsilon(131.20\sin\sigma\tau)+\ldots,$$
$$Q_{22}=5.51+\varepsilon(32.88\cos\sigma\tau)+\ldots,\qquad P_{23}=Q_{23}=R_{21}=R_{22}=0,$$
$$R_{23}=6.51+\varepsilon(32.88\cos\sigma\tau)+\ldots.$$

Equilibrium point (*c*).

$$P_{21}=-3.16-\varepsilon(6.82\cos\sigma\tau)+\ldots,\qquad P_{22}=Q_{21}=\varepsilon(6.86\sin\sigma\tau)+\ldots,$$
$$Q_{22}=0.08+\varepsilon(3.41\cos\sigma\tau)+\ldots,\qquad P_{23}=Q_{23}=R_{21}=R_{22}=0,$$
$$R_{23}=1.08+\varepsilon(3.41\cos\sigma\tau)+\ldots.$$

If we consider only the terms of (11) which are linear in p_j, q_j, and r_j, we obtain the *equations of variation.** For the orbits of Class A they are

$$\left.\begin{aligned}\ddot{p}_1-2(1+\delta_1)\dot{q}_1+(1+\delta_1)^2(P_{11}p_1+P_{12}q_1+P_{13}r_1)&=0,\\ \ddot{q}_1+2(1+\delta_1)\dot{p}_1+(1+\delta_1)^2(Q_{11}p_1+Q_{12}q_1+Q_{13}r_1)&=0,\\ \ddot{r}_1+(1+\delta_1)^2(R_{11}p_1+R_{12}q_1+R_{13}r_1)&=0,\end{aligned}\right\}\qquad(12\text{A})$$

and for orbits of Class B, they are

$$\left.\begin{aligned}\ddot{p}_2-2(1+\delta_2)\dot{q}_2+(1+\delta_2)^2(P_{21}p_2+P_{22}q_2)&=0,\\ \ddot{q}_2+2(1+\delta_2)\dot{p}_2+(1+\delta_2)^2(Q_{21}p_2+Q_{22}q_2)&=0,\\ \ddot{r}_2+(1+\delta_2)^2R_{23}r_2&=0.\end{aligned}\right\}\qquad(12\text{B})$$

* Poincaré, *loc. cit.*, Vol. I, Chap. 4.

The periodic solutions (8) and (9) are called the *generating solutions** of (12A) and (12B) respectively.

§ 6. THE SOLUTIONS OF THE EQUATIONS OF VARIATION (12A).

The equations of variation are linear differential equations having periodic coefficients, the periods being the same as those of the corresponding generating solutions. Differential equations of this type were first treated by Hill† in his celebrated memoir on the lunar theory. A very extensive list of references to the literature of these differential equations was given by Baker in a recent memoir.‡

The method which we shall adopt in the construction of the solutions of these equations is the one developed by Moulton and Macmillan.§ The form‖ of the solution is, in general

$$e^{\alpha\tau}\phi(\tau),$$

where $\phi(\tau)$ is a periodic function having the same period as the coefficients of the differential equations, and α is a constant called the *characteristic exponent.*

It has been shown by Poincaré¶ that if the generating solutions contain an arbitrary constant which does not appear explicitly in the original differential equations, then a set of solutions of the equations of variation is obtained by taking the first partial derivatives of the generating solutions with respect to this constant. The characteristic exponent associated with this set of solutions is zero, and the solutions themselves are either periodic or consist of periodic functions plus τ times other periodic functions. The initial time t_0 is one such arbitrary constant which is present in all differential equations of dynamics in which the potential function *e.g.*, U in equations (1), does not contain the time explicitly. Differentiation of the generating solutions with respect to this constant yields a set of periodic solutions. Another such constant usually present is the scale factor of the generating solutions. Differentiation of the generating solutions with respect to this constant yields a set of solutions comprised of a periodic function plus $K\tau$, K a constant, times the periodic function obtained by differentiating with respect to t_0.

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* Poincaré, *loc. cit.* Vol. I, Chap. 4.

† G. W. Hill, "Collected Works," Vol. I, p. 243; *Acta Mathematica*, Vol. VIII, pp. 1–36.

‡ H. F. Baker, "On Certain Linear Differential Equations of Astronomical Interest," *Philosophical Transactions of the Royal Society of London*, Series A (1916), Vol. 216, pp. 129–186.

§ Moulton and Macmillan, "On the Solutions of Certain Types of Linear Differential Equations with Periodic Coefficients," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIII (1911), No. 1, pp. 63–97.

‖ Floquet, *Annales Scientifiques de l'École Superieure*, Series II, Vol. XII (1883), p. 47.

¶ Poincaré, *loc cit.*, Vol. I Chap. 4.

Let us consider first equations (12A), and those solutions which have characteristic exponents different from zero. These equations are simultaneous and must be considered together. To find their solutions let

$$p_1 = K_1 e^{\alpha_1\tau} u_1, \quad q_1 = L_1 e^{\alpha_1\tau} v_1, \quad r_1 = M_1 e^{\alpha_1\tau} w_1, \tag{13}$$

where K_1, L_1, and M_1 are arbitrary constants. An existence proof, which is omitted here, would show that

$$\left.\begin{aligned}
\alpha_1 &= \alpha_1^{(0)} + \alpha_1^{(2)}\varepsilon^2 + \ldots + \alpha_1^{(2k)}\varepsilon^{2k} + \ldots,\\
u_1 &= u_1^{(0)} + u_1^{(2)}\varepsilon^2 + \ldots + u_1^{(2k)}\varepsilon^{2k} + \ldots,\\
v_1 &= v_1^{(0)} + v_1^{(2)}\varepsilon^2 + \ldots + v_1^{(2k)}\varepsilon^{2k} + \ldots,\\
w_1 &= w_1^{(1)}\varepsilon + w_1^{(3)}\varepsilon^3 + \ldots + w_1^{(2k+1)}\varepsilon^{2k+1} + \ldots,
\end{aligned}\right\} \tag{14}$$

where the $\alpha_1^{(2k)}$ are constants so determined that the various $u_1^{(2k)}$, $v_1^{(2k)}$, and $w_1^{(2k+1)}$ shall be periodic with the period $2\pi/\sqrt{A}$ in τ. On substituting (13) in (12A) and considering only the terms independent of ε, it is found that $\alpha_1^{(0)}$ must satisfy the biquadratic (7) if the arbitrary constants K_1, L_1, and M_1 are to have values other than the trivial ones $K_1 = L_1 = M_1 = 0$. It is shown in the "Oscillating Satellite," § 82, that this biquadratic admits two purely imaginary solutions, denoted by $+\sigma i$ and $-\sigma i$, and two real solutions, denoted by $+\rho$ and $-\rho$, for all values of μ such that $0 \leqq \mu \leqq \frac{1}{2}$, and for each equilibrium point (a), (b), and (c).

Since the functions u_1, v_1, and w_1 are multiplied by arbitrary constants, we may assume without loss of generality, that $u_1(0) = 1$. Hence

$$u_1^{(0)}(0) = 1, \quad u_1^{(2k)}(0) = 0 \qquad (k = 1, 2, \ldots \infty). \tag{15}$$

Then $v_1(0)$ and $w_1(0)$ can be determined from the differential equations which define v_1 and w_1.

The various solutions of (12A) may be constructed by substituting (14) and (13) in (12A), and giving to $\alpha_1^{(0)}$ the values σi, $-\sigma i$, ρ, and $-\rho$ in turn. By equating the coefficients of the various powers of ε we obtain sets of differential equations which can be integrated step by step. The arbitrary constants of integration arising at each step can be uniquely determined from the conditions (15). The various $\alpha_1^{(2k)}$ can likewise be uniquely determined by the condition that $u_1^{(2k)}$, $v_1^{(2k)}$, and $w_1^{(2k+1)}$ shall be periodic with the period $2\pi/\sqrt{A}$ in τ. From Poincaré's extension to Cauchy's theorem,* it is known that these solutions will converge for ε sufficiently small numerically and for all values of τ such that $0 \leqq \tau \leqq T$, where T is an arbitrary period chosen in advance. Obviously, we may choose the period $2\pi/\sqrt{A}$.

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* Poincaré, *loc. cit.*, Vol. I, p. 58.

Let us denote the solutions of (12A) when $\alpha_1^{(0)}=\sigma i$ by

$$p_1=e^{i\sigma_1\tau}u_{11},\quad q_1=e^{i\sigma_1\tau}iv_{11},\quad r_1=e^{i\sigma_1\tau}iw_{11}. \tag{16}$$

The arbitrary constants are omitted here but they appear in equations (25). It is found that σ_1 is a power series in ε^2 with real constant coefficients. The functions u_{11} and v_{11} are likewise power series in ε^2, but their coefficients are sums of cosines and i times sines of even multiples of $\sqrt{A}\tau$. The function w_{11} is a power series in odd powers of ε, vanishing with ε, and the coefficients are sums of cosines and i times sines of odd multiples of $\sqrt{A}\tau$. In each of these three functions, the highest multiple of $\sqrt{A}\tau$ which appears in the coefficient of ε^k is k. So far as the computation has been carried out, we have

$$u_{11}=1+(\,)\varepsilon^2+\ldots,\quad v_{11}=n+(\,)\varepsilon^2+\ldots,$$

$$w_{11}=\frac{3B}{4A-\sigma^2}\left[\frac{2}{\sigma}\cos\sqrt{A}\tau-\frac{i}{\sqrt{A}}\sin\sqrt{A}\tau\right]\varepsilon+(\,)\varepsilon^3+\ldots,$$

$$\sigma_1=\sigma+\varepsilon^2\left[\frac{9B^2\{\sigma^2(1-13A)-(3+7A-22A^2)\}}{16A\sigma^3(4A-\sigma^2)}-\frac{\delta_1^{(2)}(1-A)(1+2A)}{\sigma^3}\right]+\ldots..$$

Since the differential equations (12A) do not contain i, they are unaltered by a change in the sign of i. If then we change the sign of i in the differential equations and in the initial conditions (15), we obtain solutions which differ from (16) only in the sign of i. Let this set of solutions be denoted by

$$p_1=e^{-i\sigma\tau}u_{12},\quad q_1=-e^{-i\sigma\tau}iv_{12},\quad r_1=-e^{-i\sigma\tau}iw_{12}, \tag{17}$$

where u_{12}, v_{12}, and w_{12} differ from u_{11}, v_{11}, and w_{11} respectively only in the sign of i. The same result would be obtained by putting $\alpha_1^{(0)}=-i\sigma$ and proceeding as in the construction of the solution (16).

Likewise by putting $\alpha_1^{(0)}$ equal, first to ρ, and then to $-\rho$, we obtain the respective solutions

$$p_1=e^{\rho_1\tau}u_{13},\qquad q_1=e^{\rho_1\tau}v_{13},\qquad r_1=e^{\rho_1\tau}w_{13}, \tag{18}$$

and

$$p_1=e^{-\rho_1\tau}u_{14},\quad q_1=-e^{-\rho_1\tau}v_{14},\quad r_1=-e^{-\rho_1\tau}w_{14}, \tag{19}$$

where ρ_1 is a power series in ε^2 with real constant coefficients. Further u_{13}, u_{14}; v_{13}, v_{14}; w_{13}, w_{14} are similar to u_{11}, v_{11}, and w_{11} respectively, except that the coefficients of the former functions are all real; also

$$u_{14}(\tau)=u_{13}(-\tau),\quad v_{14}(\tau)=v_{13}(-\tau),\quad w_{14}(\tau)=w_{13}(-\tau).$$

The computation for the first terms of these series gives

$$u_{13}=1+(\,)\varepsilon^2+\ldots,\quad v_{13}=m+(\,)\varepsilon^2+\ldots,$$

$$w_{13}=-\frac{3B}{4A+\rho^2}\left[\frac{2}{\rho}\cos\sqrt{A}\tau-\frac{1}{\sqrt{A}}\sin\sqrt{A}\tau\right]\varepsilon+(\,)\varepsilon^3+\ldots,$$

$$\rho_1=\rho+\varepsilon^2\left[\frac{9B^2\{\rho^2(1-13A)+3+7A-22A^2\}}{14A\rho^3(4A+\rho^2)}+\frac{\delta_1^{(2)}(1-A)(1+2A)}{\rho^3}\right]+\ldots..$$

Obviously, the solutions (19) differ from (18) only in the sign of τ, q_1, and r_1. This property can be derived directly from the differential equations, which we proceed to show. We shall show that if

$$p_j = F_{j1}(\tau), \quad q_j = F_{j2}(\tau), \quad r_j = F_{j3}(\tau) \tag{20}$$

are solutions of the differential equations (11), then

$$p_j = F_{j1}(-\tau), \quad q_j = -F_{j2}(-\tau), \quad r_j = -F_{j3}(-\tau)$$

are likewise solutions. If this property holds for equations (11) it will evidently hold for equations (12A) or (12B), *i. e.*, when the right members of (11) are omitted.

Let us consider the differential equations (1). Since U is even in η and ζ, it follows that $\frac{\partial U}{\partial \xi}$ is even in η and ζ, $\frac{\partial U}{\partial \eta}$ is odd in η and even in ζ, and $\frac{\partial U}{\partial \zeta}$ is even in η and odd in ζ. After the substitutions (2) are made, the right members of (3) are functions of $\bar{y}$ and $\bar{z}$ as indicated. As the substitutions (4) do not alter the parity of y and z, then it follows that the right members of (5) have the same parity in y and z that the corresponding right members have in $\bar{y}$ and $\bar{z}$. When the substitutions (10) are made in (5) we obtain (11), and the terms of (11) which carry $(1+\delta_j)^2$ as a factor in each equation arise from the corresponding right members of (5). Let these terms be denoted by (11, 1), (11, 2), and (11, 3) respectively. Then they have the following properties:

(11, 1) is even in $\{y_j, q_j\}$, and even in $\{z_j, r_j\}$,
(11, 2) is odd in $\{y_j, q_j\}$, and even in $\{z_j, r_j\}$,
(11, 3) is even in $\{y_j, q_j\}$, and odd in $\{z_j, r_j\}$,

where the braces $\{\}$ denote that the variables within are to be considered together. Further, since $\bar{y}$ is a factor of the right member of the second equation in (3), it follows that (11, 2) carries a factor which is odd in $\{y_j, q_j\}$. Similarly, since $\bar{z}$ is a factor of the right members of the third equation in (3), then (11, 3) carries a factor which is odd in $\{z_j, q_j\}$.

Now, if the signs of y_j, z_j, q_j, and r_j be changed in the above expressions, then (11, 1) remains unchanged while (11, 2) and (11, 3) change signs. On examining the generating solutions (8) and (9) we observe that changing the signs of y_j and z_j is equivalent to changing the sign of τ in these solutions. Thus, if we change the signs of τ, q_j, and r_j, (11, 1) remains unchanged while (11, 2) and (11, 3) change signs. If these changes are made in all the terms of (11), the first equation remains unchanged while each term of the remaining equations changes sign and the factor -1 can be divided out. Thus the equations (11) remain unchanged if we change the signs of τ, q_j and r_j.

Suppose now that (20) is a set of solutions of (11). Let

$$\bar{p}_j = p_j, \quad \bar{q}_j = -q_j, \quad \bar{r}_j = -r_j, \quad \bar{\tau} = -\tau. \tag{21}$$

Then the differential equations obtained by making these substitutions in (11) are identically the same in $\bar{p}_j$, $\bar{q}_j$, $\bar{r}_j$, and $\bar{\tau}$ as (11) are in the former variables. With the same initial conditions for $\bar{p}_j$, $\bar{q}_j$, $\bar{r}_j$, and $\bar{\tau}$ as for equations (11), we obtain the solutions

$$\bar{p}_j = F_{j1}(\bar{\tau}), \quad \bar{q}_j = F_{j2}(\bar{\tau}), \quad \bar{r}_j = F_{j3}(\bar{\tau}),$$

where F_{j1}, F_{j2}, and F_{j3} are the same functions as in (20). On restoring the former variables by (21) we have

$$p_j = F_{j1}(-\tau), \quad q_j = -F_{j2}(-\tau), \quad r_j = -F_{j3}(-\tau)$$

as solutions of (11). Therefore if a set of solutions of (11) is known, another set can be obtained by changing the signs of τ, q_j, and r_j in the former set.

Let us next consider the two remaining sets of solutions of (12A). These solutions have characteristic exponents zero and may be obtained, as already referred to in this section, by differentiating the generating solutions (8) with respect to the arbitrary constants t_0 and ε. Taking the constant t_0 first, we have the solutions

$$p_1 = \frac{\partial \bar{x}_1}{\partial t_0} = \frac{\partial \bar{x}_1}{\partial \tau}\frac{\partial \tau}{\partial t_0} = -\frac{1}{1+\delta_1}\left[(-2b_1\sqrt{A}\sin 2\sqrt{A}\tau)\varepsilon^2 + ()\varepsilon^4 + \dots\right]$$
$$= -\frac{1}{1+\delta_1} u_{15},$$

$$q_1 = \frac{\partial \bar{y}_1}{\partial t_0} = \frac{\partial \bar{y}_1}{\partial \tau}\frac{\partial \tau}{\partial t_0} = -\frac{1}{1+\delta_1}\left[(2c_1\sqrt{A}\cos 2\sqrt{A}\tau)\varepsilon^2 + ()\varepsilon^4 + \dots\right]$$
$$= -\frac{1}{1+\delta_1} v_{15},$$

$$r_1 = \frac{\partial \bar{z}_1}{\partial t_0} = \frac{\partial \bar{z}_1}{\partial \tau}\frac{\partial \tau}{\partial t_0} = -\frac{1}{1+\delta_1}\left[(\cos\sqrt{A}\tau)\varepsilon + ()\varepsilon^3 + \dots\right] = -\frac{1}{1+\delta_1} w_{15}.$$

As these solutions are later multiplied by arbitrary constants, we may neglect the factor $-\dfrac{1}{1+\delta_1}$ and consider

$$p_1 = u_{15}, \quad q_1 = v_{15}, \quad r_1 = w_{15} \tag{22}$$

as the solutions. Similarly, on differentiating (8) with respect to ε we obtain

$$p_1 = \frac{\partial \bar{x}_1}{\partial \varepsilon}, \quad q_1 = \frac{\partial \bar{y}_1}{\partial \varepsilon}, \quad r_1 = \frac{\partial \bar{z}_1}{\partial \varepsilon}.$$

Since $\tau = (t - t_0)/(1+\delta_1)$ and δ_1 is a function of ε, then the constant ε enters

the generating solutions not only explicitly but implicitly through δ_1 and τ. Hence

$$\frac{\partial \bar{x}_1}{\partial \varepsilon}=\left(\frac{\partial \bar{x}_1}{\partial \varepsilon}\right)+\frac{\partial \bar{x}_1}{\partial \tau}\frac{\partial \tau}{\partial \delta_1}\frac{\partial \delta_1}{\partial \varepsilon},\quad \frac{\partial \bar{y}_1}{\partial \varepsilon}=\left(\frac{\partial \bar{y}_1}{\partial \varepsilon}\right)+\frac{\partial \bar{y}_1}{\partial \tau}\frac{\partial \tau}{\partial \delta_1}\frac{\partial \delta_1}{\partial \varepsilon},$$
$$\frac{\partial \bar{z}_1}{\partial \varepsilon}=\left(\frac{\partial \bar{z}_1}{\partial \varepsilon}\right)+\frac{\partial \bar{z}_1}{\partial \tau}\frac{\partial \tau}{\partial \delta_1}\frac{\partial \delta_1}{\partial \varepsilon},$$

where the parentheses around the partial derivatives denote that the differentiation is performed only in so far as ε occurs explicitly. When the differentiations are performed, we have

$$p_1=u_{16}+K\tau u_{15},\quad q_1=v_{16}+K\tau v_{15},\quad r_1=w_{16}+K\tau w_{15},\tag{23}$$

where

$$u_{16}=\left(\frac{\partial \bar{x}_1}{\partial \varepsilon}\right)=2(a_1+b_1\cos 2\sqrt{A}\tau)\varepsilon+(\,)\varepsilon^3+\dots,$$
$$v_{16}=\left(\frac{\partial \bar{y}_1}{\partial \varepsilon}\right)=2(c_1\sin 2\sqrt{A}\tau)\varepsilon+(\,)\varepsilon^3+\dots,$$
$$w_{16}=\left(\frac{\partial \bar{z}_1}{\partial \varepsilon}\right)=\frac{1}{\sqrt{A}}\sin\sqrt{A}\tau+(\,)\varepsilon^2+\dots,$$
$$K=-\frac{1}{1+\delta_1}\,[2\delta_1^{(2)}\varepsilon+(\,)\varepsilon^3+\dots].$$

It remains now to show that the solutions which we have obtained, viz., (16), (17), (18), (19), (22), and (23) constitute a fundamental set. The criterion for a fundamental set is that the determinant formed from these solutions and their first derivatives with respect to τ shall be different from zero. This determinant is

$$\begin{vmatrix}
e^{i\sigma_1\tau}u_{11}, & e^{-i\sigma_1\tau}u_{12}, & e^{\rho_1\tau}u_{13}, & e^{-\rho_1\tau}u_{14}, & u_{15},\ u_{16}+K\tau u_{15},\\
e^{i\sigma_1\tau}(i\sigma_1u_{11}+\dot{u}_{11}), & e^{-i\sigma_1\tau}(-i\sigma_1u_{12}+\dot{u}_{12}), & e^{\rho_1\tau}(\rho_1u_{13}+\dot{u}_{13}), & e^{-\rho_1\tau}(-\rho_1u_{14}+\dot{u}_{14}), & \dot{u}_{15},\ \dot{u}_{16}+K(\tau\dot{u}_{15}+u_{15}),\\
e^{i\sigma_1\tau}iv_{11}, & e^{-i\sigma_1\tau}(-iv_{12}), & e^{\rho_1\tau}v_{13}, & e^{-\rho_1\tau}(-v_{14}), & v_{15},\ v_{16}+K\tau v_{15},\\
e^{i\sigma_1\tau}(-\sigma_1v_{11}+i\dot{v}_{11}), & e^{-i\sigma_1\tau}(\sigma_1v_{12}-i\dot{v}_{12}), & e^{\rho_1\tau}(\rho_1v_{13}+\dot{v}_{13}), & e^{-\rho_1\tau}(\rho_1v_{14}-\dot{v}_{14}), & \dot{v}_{15},\ \dot{v}_{16}+K(\tau\dot{v}_{15}+v_{15}),\\
e^{i\sigma_1\tau}iw_{11}, & e^{-i\sigma_1\tau}(-iw_{12}), & e^{\rho_1\tau}w_{13}, & e^{-\rho_1\tau}(-w_{14}), & w_{15},\ w_{16}+K\tau w_{15},\\
e^{i\sigma_1\tau}(-\sigma_1w_{11}+i\dot{w}_{11}), & e^{-i\sigma_1\tau}(\sigma_1w_{12}-i\dot{w}_{12}), & e^{\rho_1\tau}(\rho_1w_{13}+\dot{w}_{13}), & e^{-\rho_1\tau}(\rho_1w_{14}-\dot{w}_{14}), & \dot{w}_{15},\ \dot{w}_{16}+K(\tau\dot{w}_{15}+w_{15}).
\end{vmatrix}\tag{24}$$

It is a constant,* and its value can be determined with the least difficulty by putting $\tau=0$. Thus we obtain

$$\Delta_1=4i\varepsilon(m\rho+n\sigma)(m\sigma-n\rho)+\text{ terms of higher degree in }\varepsilon.$$

It is shown in the "Oscillating Satellite," equations (36), that $m\rho+n\sigma$ and $m\sigma-n\rho$ are different from zero. Hence the determinant Δ_1 is different from

* Moulton, "Periodic Orbits," Chap. 1, Sec. 18.

zero for ε not zero, but sufficiently small numerically. Therefore the solutions which have been obtained for (12A) constitute a fundamental set, and the most general solutions are

$$\left.\begin{aligned} p_1&=K_1e^{i\sigma_1\tau}u_{11}+K_2e^{-i\sigma_1\tau}u_{12}+K_3e^{\rho_1\tau}u_{13}+K_4e^{-\rho_1\tau}u_{14}+K_5u_{15}\\ &\qquad+K_6(u_{16}+K\tau u_{15}),\\ q_1&=K_1e^{i\sigma_1\tau}iv_{11}-K_2e^{-i\sigma_1\tau}iv_{12}+K_3e^{\rho_1\tau}v_{13}-K_4e^{-\rho_1\tau}v_{14}+K_5v_{15}\\ &\qquad+K_6(v_{16}+K\tau v_{15}),\\ r_1&=K_1e^{i\sigma_1\tau}iw_{11}-K_2e^{-i\sigma_1\tau}iw_{12}+K_3e^{\rho_1\tau}w_{13}-K_4e^{-\rho_1\tau}w_{14}+K_5w_{15}\\ &\qquad+K_6(w_{16}+K\tau w_{15}),\end{aligned}\right\} \quad (25)$$

where $K_1, \ldots, K_6$ are arbitrary constants.

§7. The Solutions of the Equations of Variation (12B).

The solutions of the equations (12B) are obtained in the same way as the preceding solutions were found. The construction is simplified by the fact that the last equation is independent of the first two.

The two sets of solutions of the first two equations of (12B) which have characteristic exponents different from zero, are

$$p_2=e^{\rho_2\tau}u_{21}, \quad q_2=e^{\rho_2\tau}v_{21}, \tag{26}$$

and

$$p_2=e^{-\rho_2\tau}u_{22}, \quad q_2=-e^{-\rho_2\tau}v_{22}, \tag{27}$$

where ρ_2 is a power series in ε with real coefficients, the terms of lower degrees being

$$\rho_2=\rho+0\varepsilon+\rho_2^{(2)}\varepsilon^2+\rho_2^{(3)}\varepsilon^3+\ldots.$$

The functions u_{21}, v_{21}, u_{22}, and v_{22} are power series in ε with sums of sines and cosines of multiples of $\sigma\tau$ in the coefficients, the highest multiple of $\sigma\tau$ which occurs in the coefficient of ε^k being k. All the numerical coefficients in these series are real. From the property of the differential equations (11) which was proved in the preceding section, it follows that the solutions (27) are obtained from (26) by changing the signs of τ and q_2. Thus u_{22} and v_{22} differ from u_{21} and v_{21} respectively only in the signs of the sines.

Since these solutions (26) and (27) are multiplied later by arbitrary constants, we may take $u_{21}(0)=1$. The algebraic forms of the functions u_{21} and v_{21} are

$$\begin{aligned} u_{21}&=1+(a_{21}\cos\sigma\tau+b_{21}\sin\sigma\tau)\varepsilon+(\;)\varepsilon^2+\ldots,\\ v_{21}&=m+(a_{22}\cos\sigma\tau+b_{22}\sin\sigma\tau)\varepsilon+(\;)\varepsilon^2+\ldots,\\ u_{22}(\tau)&=u_{21}(-\tau), \quad v_{22}(\tau)=v_{21}(-\tau).\end{aligned}$$

The two remaining solutions of the first two equations of (12B) are obtained by differentiating the generating solutions (9) with respect to t_0 and ε. Differentiation with respect to t_0 gives

$$p_2=\frac{\partial\bar{x}_2}{\partial t_0}=-\frac{1}{1+\delta_2}\frac{\partial\bar{x}_2}{\partial\tau},\quad q_2=\frac{\partial\bar{y}_2}{\partial t_0}=-\frac{1}{1+\delta_2}\frac{\partial\bar{y}_2}{\partial\tau},$$

and, since the multiplier $-\dfrac{1}{1+\delta_2}$ may be dropped, the solutions become

$$\left.\begin{aligned}p_2=u_{23}=\frac{\partial\bar{x}_2}{\partial\tau}&=(-\sigma\sin\sigma\tau)\varepsilon-(b_2\sigma\sin\sigma\tau+2c_2\sigma\sin 2\sigma\tau)\varepsilon^2+\ldots,\\ q_2=v_{23}=\frac{\partial\bar{y}_2}{\partial\tau}&=(-n\sigma\cos\sigma\tau)\varepsilon-(nb_2\sigma\cos\sigma\tau-2d_2\sigma\cos 2\sigma\tau)\varepsilon^2+\ldots.\end{aligned}\right\}\quad(28)$$

Differentiation with respect to ε gives

$$p_2=u_{24}+L\tau u_{23},\quad q_2=v_{24}+L\tau v_{23},\qquad(29)$$

where

$$\begin{aligned}u_{24}&=\left(\frac{\partial\bar{x}_2}{\partial\varepsilon}\right)=\cos\sigma\tau+2(a_2+b_2\cos\sigma\tau+c_2\cos 2\sigma\tau)\varepsilon+\ldots,\\ v_{24}&=\left(\frac{\partial\bar{y}_2}{\partial\varepsilon}\right)=-n\sin\sigma\tau-2(nb_2\sin\sigma\tau-d_2\sin 2\sigma\tau)\varepsilon+\ldots,\\ L&=-\frac{1}{1+\delta_2}(2\delta_2^{(2)}\varepsilon+3\delta_2^{(3)}\varepsilon^2+\ldots).\end{aligned}$$

The determinant of these solutions (26), (27), (28), and (29) and their first derivatives with respect to τ is

$$\Delta_2=\begin{vmatrix}e^{\rho_2\tau}u_{21}, & e^{-\rho_2\tau}u_{22}, & u_{23}, & u_{24}+L\tau u_{23},\\ e^{\rho_2\tau}(\rho_2u_{21}+\dot{u}_{21}), & e^{-\rho_2\tau}(-\rho_2u_{22}+\dot{u}_{22}), & \dot{u}_{23}, & \dot{u}_{24}+L(\tau\dot{u}_{23}+u_{23}),\\ e^{\rho_2\tau}v_{21}, & e^{-\rho_2\tau}(-v_{22}), & v_{23}, & v_{24}+L\tau v_{23},\\ e^{\rho_2\tau}(\rho_2v_{21}+\dot{v}_{21}), & e^{-\rho_2\tau}(\rho_2v_{22}-\dot{v}_{22}), & \dot{v}_{23}, & \dot{v}_{24}+L(\tau\dot{v}_{23}+v_{23}),\end{vmatrix}\quad(30)$$

$$=2\sigma\varepsilon(m\rho+n\sigma)(m\sigma-n\rho)+\text{terms of higher degree in }\varepsilon.$$

As in the former determinant Δ_1, equation (24), the factors $m\rho+n\sigma$ and $m\sigma-n\rho$ are different from zero, and since $\sigma\neq0$ it follows that $\Delta_2\neq0$ for ε not zero, but sufficiently small numerically. Hence the solutions (26), (27), (28), and (29) constitute a fundamental set of solutions of the first two equations, and the most general solutions are

$$\left.\begin{aligned}p_2&=L_1e^{\rho_2\tau}u_{21}+L_2e^{-\rho_2\tau}u_{22}+L_3u_{23}+L_4(u_{24}+L\tau u_{23}),\\ q_2&=L_1e^{\rho_2\tau}v_{21}-L_2e^{-\rho_2\tau}v_{22}+L_3v_{23}+L_4(v_{24}+L\tau v_{23}),\end{aligned}\right\}\quad(31)$$

where $L_1,\ldots,L_4$ are arbitrary constants.

The general solution of the last equation of (12B) is readily obtained, and it has the form

$$r_2 = L_5 e^{i\omega\tau} w_{21} + L_6 e^{-i\omega\tau} w_{22}, \tag{32}$$

where L_5 and L_6 are arbitrary constants, and ω, w_{21}, and w_{22} are power series in ε of the form

$$w_{21} = 1 + \frac{3B\varepsilon}{\sigma^2(\sigma^2-4A)}(\sigma^2 \cos\sigma\tau - 2i\sqrt{A}\sin\sigma\tau - \sigma^2) + \ldots,$$

$$w_{22} = 1 + \frac{3B\varepsilon}{\sigma^2(\sigma^2-4A)}(\sigma^2 \cos\sigma\tau + 2i\sqrt{A}\sin\sigma\tau - \sigma^2) + \ldots,$$

$$\omega = \sqrt{A} + 0\varepsilon + \omega_2\varepsilon^2 + \ldots.$$

Since L_5 and L_6 are arbitrary, the initial values of w_{21} and w_{22} may be chosen so that $w_{21}(0) = w_{22}(0) = 1$. The coefficients of the various powers of ε in these functions are sums of cosines and i times sines of multiples of $\sigma\tau$, the highest multiple which occurs in the coefficient of ε^k being k. Further, w_{21} and w_{22} differ only in the sign of i. In the series for ω, all the coefficients are real. The determinant of the two solutions in (32) and their derivatives with respect to τ is

$$\Delta_3 = \begin{vmatrix} e^{i\omega\tau} w_{21}, & e^{-i\omega\tau} w_{22}, \\ e^{i\omega\tau}(i\omega w_{21} + \dot{w}_{21}), & e^{-i\omega\tau}(-i\omega w_{21} + \dot{w}_{21}), \end{vmatrix}$$

$$= -2i\omega + \text{terms in } \varepsilon,$$

and this is different from zero for $|\varepsilon|$ sufficiently small. Hence the solutions (32) constitute a fundamental set of solutions of the last equation of (12B).

§ 8. THE CONSTRUCTION OF ASYMPTOTIC SOLUTIONS.

Having determined the solutions of the equations of variation, we shall now proceed to construct the asymptotic solutions of (11). We shall consider first the solutions which approach zero as τ approaches $+\infty$, and then show how to obtain from these the solutions which approach zero as τ approaches $-\infty$.

In making the constructions, it is convenient to introduce a parameter γ by the substitutions

$$p_j = \bar{p}_j\gamma, \quad q_j = \bar{q}_j\gamma, \quad r_j = \bar{r}_j\gamma, \tag{33}$$

where $\bar{p}_j$, $\bar{q}_j$, and $\bar{r}_j$ are new dependent variables. Then, since the asymptotic solutions are to be constructed as power series in γ, we put

$$\left.\begin{aligned} \bar{p}_j = p_j^{(0)} + p_j^{(1)}\gamma + p_j^{(2)}\gamma^2 + \ldots, \quad \bar{q}_j = q_j^{(0)} + q_j^{(1)}\gamma + q_j^{(2)}\gamma^2 + \ldots, \\ \bar{r}_j = r_j^{(0)} + r_j^{(1)}\gamma + r_j^{(2)}\gamma^2 + \ldots. \end{aligned}\right\} \tag{34}$$

When (34) and (33) are substituted in (11), the factor γ will divide out. Let the resulting differential equations be denoted by (11′). On equating the coefficients of the various powers of γ in these equations, we obtain sets of differential equations which define the functions $p_j^{(k)}$, $q_j^{(k)}$, and $r_j^{(k)}$ in (34). In order that the solutions shall be asymptotic we must impose the condition

(C_1), that each term of the solutions for the various $p_j^{(k)}$, $q_j^{(k)}$, and $r_j^{(k)}$ shall contain $e^{-\rho_j\tau}$ as a factor.

As an arbitrary constant arises at each step of the integration we may impose the further condition

(C_2), that $p_j(0)=\alpha_j$,

from which it follows that

$$p_j^{(0)}(0)=\alpha_j, \quad p_j^{(k)}(0)=0, \quad (k=1, \ldots . \infty).$$

Asymptotic Orbits of Class A.

Let us consider first the orbits which are asymptotic to the periodic orbits of Class A. We put $j=1$ and consider the various differential equations obtained by equating the coefficients of the same powers of γ in (11′).

For the terms in $p_1^{(0)}$, $q_1^{(0)}$, and $r_1^{(0)}$ we obtain a set of differential equations which are the same as (12A), except for the superscript 0. The general solutions are the same as (25), and when the condition (C_1) is imposed all the constants of integration must be zero except the one associated with $e^{-\rho_1\tau}$. From condition (C_2) it follows that this constant must have the value α_1. Therefore the desired solutions are

$$p_1^{(0)}=\alpha_1 e^{-\rho_1\tau}u_{14}, \quad q_1^{(0)}=-\alpha_1 e^{-\rho_1\tau}v_{14}, \quad r_1^{(0)}=-\alpha_1 e^{-\rho_1\tau}w_{14}. \tag{35}$$

The differential equations which are obtained from equating the coefficients of γ to the first degree in (11′) have the same left members as (12A), except for the superscript 1 on the variables. On denoting the corresponding right members by $P_1^{(1)}$, $Q_1^{(1)}$, and $R_1^{(1)}$, we find that

$$P_1^{(1)}=\varepsilon\alpha_1^2 e^{-2\rho_1\tau}U_{12}^{(1)}, \quad Q_1^{(1)}=\varepsilon\alpha_1^2 e^{-2\rho_1\tau}V_{12}^{(1)}, \quad R_1^{(1)}=\varepsilon\alpha_1^2 e^{-2\rho_1\tau}W_{12}^{(1)},$$

where $U_{12}^{(1)}$, $V_{12}^{(1)}$, and $W_{12}^{(1)}$ are power series of the same form as u_{14}, v_{14}, and w_{14} respectively. The complementary functions of these differential equations are the same as (25), but let us suppose that the arbitrary multipliers are

$k_1^{(1)}, \ldots, k_6^{(1)}$. The particular integrals can be obtained by varying these multipliers. Thus

$$\left.\begin{aligned}
&\dot{k}_1^{(1)} e^{i\sigma_1\tau} u_{11} + \dot{k}_2^{(1)} e^{-i\sigma_1\tau} u_{12} + \dot{k}_3^{(1)} e^{\rho_1\tau} u_{13} + \dot{k}_4^{(1)} e^{-\rho_1\tau} u_{14} + \dot{k}_5^{(1)} u_{15} \\
&\qquad\qquad + \dot{k}_6^{(1)} (u_{16} + K\tau u_{15}) = 0, \\
&\dot{k}_1^{(1)} e^{i\sigma_1\tau} (i\sigma_1 u_{11} + \dot{u}_{11}) - \dot{k}_2^{(1)} e^{-i\sigma_1\tau} (i\sigma_1 u_{12} - \dot{u}_{12}) + \dot{k}_3^{(1)} e^{\rho_1\tau} (\rho_1 u_{13} + \dot{u}_{13}) \\
&\qquad - \dot{k}_4^{(1)} e^{-\rho_1\tau} (\rho_1 u_{14} - \dot{u}_{14}) + \dot{k}_5^{(1)} \dot{u}_{15} + \dot{k}_6^{(1)} [\dot{u}_{16} + K(u_{15} + \tau\dot{u}_{15})] = P_1^{(1)}, \\
&\dot{k}_1^{(1)} e^{i\sigma_1\tau} i v_{11} - \dot{k}_2^{(1)} e^{-i\sigma_1\tau} i v_{12} + \dot{k}_3^{(1)} e^{\rho_1\tau} v_{13} - \dot{k}_4^{(1)} e^{-\rho_1\tau} v_{14} + \dot{k}_5^{(1)} v_{15} \\
&\qquad\qquad + \dot{k}_6^{(1)} (v_{16} + K\tau v_{15}) = 0, \\
&\dot{k}_1^{(1)} e^{i\sigma_1\tau} (-\sigma_1 v_{11} + i\dot{v}_{11}) + \dot{k}_2^{(1)} e^{-i\sigma_1\tau} (\sigma_1 v_{12} - i\dot{v}_{12}) + \dot{k}_3^{(1)} e^{\rho_1\tau} (\rho_1 v_{13} + \dot{v}_{13}) \\
&\qquad + \dot{k}_4^{(1)} e^{-\rho_1\tau} (\rho_1 v_{14} - \dot{v}_{14}) + \dot{k}_5^{(1)} \dot{v}_{15} + \dot{k}_6^{(1)} [\dot{v}_{16} + K(v_{15} + \tau\dot{v}_{15})] = Q_1^{(1)}, \\
&\dot{k}_1^{(1)} e^{i\sigma_1\tau} i w_{11} - \dot{k}_2^{(1)} e^{-i\sigma_1\tau} i w_{12} + \dot{k}_3^{(1)} e^{\rho_1\tau} w_{13} - \dot{k}_4^{(1)} e^{-\rho_1\tau} w_{14} + \dot{k}_5^{(1)} w_{15} \\
&\qquad\qquad + \dot{k}_6^{(1)} (w_{16} + K\tau w_{15}) = 0, \\
&\dot{k}_1^{(1)} e^{i\sigma_1\tau} (-\sigma_1 w_{11} + i\dot{w}_{11}) + \dot{k}_2^{(1)} e^{-i\sigma_1\tau} (\sigma_1 w_{12} - i\dot{w}_{12}) + \dot{k}_3^{(1)} e^{\rho_1\tau} (\rho_1 w_{13} + \dot{w}_{13}) \\
&\qquad + \dot{k}_4^{(1)} e^{-\rho_1\tau} (\rho_1 w_{14} - \dot{w}_{14}) + \dot{k}_5^{(1)} \dot{w}_{15} + \dot{k}_6^{(1)} [\dot{w}_{16} + K(w_{15} + \tau\dot{w}_{15})] = R_1^{(1)}.
\end{aligned}\right\} \quad (36)$$

The determinant of the coefficients of $\dot{k}_1^{(1)}, \ldots, \dot{k}_6^{(1)}$ is Δ_1, equation (24), and since it is different from zero for ε not zero but sufficiently small numerically these equations can be solved for $\dot{k}_1^{(1)}, \ldots, \dot{k}_6^{(1)}$. Thus

$$\dot{k}_l^{(1)} = \frac{\Delta_{1l}^{(1)}}{\Delta_1}, \quad (l = 1, \ldots, 6), \qquad (37)$$

where $\Delta_{1l}^{(1)}$ is the determinant formed by replacing the elements of the l-th column of Δ_1, with 0, $P_1^{(1)}$, 0, $Q_1^{(1)}$, 0, and $R_1^{(1)}$ respectively. Since $P_1^{(1)}$, $Q_1^{(1)}$, and $R_1^{(1)}$ contain no terms in $e^{\pm i\sigma_1\tau}$ or $e^{\pm\rho_1\tau}$, the integrations of (37) for $k_1^{(1)}, \ldots, k_4^{(1)}$ will yield no terms in τ explicitly. Such terms will occur, however, in the integrations for $k_5^{(1)}$ and $k_6^{(1)}$, but when they are substituted in the complementary functions they will cancel off. The complete solutions are thus found to be

$$\left.\begin{aligned}
p_1^{(1)} &= K_1^{(1)} e^{i\sigma_1\tau} u_{11} + K_2^{(1)} e^{-i\sigma_1\tau} u_{12} + K_3^{(1)} e^{\rho_1\tau} u_{13} + K_4^{(1)} e^{-\rho_1\tau} u_{14} + K_5^{(1)} u_{15} \\
&\qquad + K_6^{(1)} (u_{16} + K\tau u_{15}) + \varepsilon\alpha_1^2 e^{-2\rho_1\tau} u_{12}^{(1)}, \\
q_1^{(1)} &= K_1^{(1)} e^{i\sigma_1\tau} i v_{11} - K_2^{(1)} e^{-i\sigma_1\tau} i v_{12} + K_3^{(1)} e^{\rho_1\tau} v_{13} - K_4^{(1)} e^{-\rho_1\tau} v_{14} + K_5^{(1)} v_{15} \\
&\qquad + K_6^{(1)} (v_{16} + K\tau v_{15}) + \varepsilon\alpha_1^2 e^{-2\rho_1\tau} v_{12}^{(1)}, \\
r_1^{(1)} &= K_1^{(1)} e^{i\sigma_1\tau} i w_{11} - K_2^{(1)} e^{-i\sigma_1\tau} i w_{12} + K_3^{(1)} e^{\rho_1\tau} w_{13} - K_4^{(1)} e^{-\rho_1\tau} w_{14} + K_5^{(1)} w_{15} \\
&\qquad + K_6^{(1)} (w_{16} + K\tau w_{15}) + \varepsilon\alpha_1^2 e^{-2\rho_1\tau} w_{12}^{(1)},
\end{aligned}\right\} \quad (38)$$

where $K_1^{(1)}, \ldots, K_6^{(1)}$ are the constants of integration, and $u_{12}^{(1)}$, $v_{12}^{(1)}$, and $w_{12}^{(1)}$ are functions similar to u_{14}, v_{14}, and w_{14} respectively. On imposing condition (C_1) upon these solutions we obtain

$$K_1^{(1)} = K_2^{(1)} = K_3^{(1)} = K_5^{(1)} = K_6^{(1)} = 0,$$

and from condition (C_2) we have the remaining constant

$$K_4^{(1)} = -\varepsilon\alpha_1^2 u_{12}^{(1)}(0).$$

When the constants of integration are thus determined, the solutions (38) take the form

$$\left.\begin{aligned} p_1^{(1)} = \varepsilon\alpha_1^2(e^{-\rho_1\tau}u_{11}^{(1)} + e^{-2\rho_1\tau}u_{12}^{(1)}), \quad q_1^{(1)} = \varepsilon\alpha_1^2(e^{-\rho_1\tau}v_{11}^{(1)} + e^{-2\rho_1\tau}v_{12}^{(1)}),\\ r_1^{(1)} = \varepsilon\alpha_1^2(e^{-\rho_1\tau}w_{11}^{(1)} + e^{-2\rho_1\tau}w_{12}^{(1)}), \end{aligned}\right\} \tag{39}$$

where the functions $u_{11}^{(1)}, u_{12}^{(1)}$; $v_{11}^{(1)}, v_{12}^{(1)}$; $w_{11}^{(1)}, w_{12}^{(1)}$ are of the same form as u_{14}, v_{14}, and w_{14} respectively.

The remaining steps of the integration can be carried on in essentially the same way. By an induction to the general term we shall show that the solutions of (11) can be constructed to any desired degree of accuracy.

Let us assume that $p_1^{(\nu)}, q_1^{(\nu)}, r_1^{(\nu)}$ have been constructed for $\nu = 1, 2, \ldots, k-1$, and that

$$p_1^{(\nu)} = \varepsilon^\nu\alpha_1^{\nu+1}\sum_{l=1}^{\nu+1} e^{-l\rho_1\tau}u_{1l}^{(\nu)}, \quad q_1^{(\nu)} = \varepsilon^\nu\alpha_1^{\nu+1}\sum_{l=1}^{\nu+1} e^{-l\rho_1\tau}v_{1l}^{(\nu)}, \quad r_1^{(\nu)} = \varepsilon^\nu\alpha_1^{\nu+1}\sum_{l=1}^{\nu+1} e^{-l\rho_1\tau}w_{1l}^{(\nu)}, \tag{40}$$

where the functions $u_{1l}^{(\nu)}$, $v_{1l}^{(\nu)}$, and $w_{1l}^{(\nu)}$ are similar to u_{14}, v_{14}, and w_{14} respectively. We propose to show that the solutions for $p_1^{(k)}$, $q_1^{(k)}$, and $r_1^{(k)}$ are the same as (40) if $\nu = k$.

Consider the set of differential equations obtained by equating the coefficients of γ^k in (11′). The left members are the same as (12A), except for the superscript k on p_1, q_1, and r_1. Let the corresponding right members be denoted by $P_1^{(k)}$, $Q_1^{(k)}$, and $R_1^{(k)}$. Then it is found that

$$P_1^{(k)} = \varepsilon^k\alpha_1^{k+1}\sum_{l=1}^{k+1} e^{-l\rho_1\tau}U_{1l}^{(k)}, \quad Q_1^{(k)} = \varepsilon^k\alpha_1^{k+1}\sum_{l=1}^{k+1} e^{-l\rho_1\tau}V_{1l}^{(k)}, \quad R_1^{(k)} = \varepsilon^k\alpha_1^{k+1}\sum_{l=1}^{k+1} e^{-l\rho_1\tau}W_{1l}^{(k)},$$

where $U_{1l}^{(k)}$, $V_{1l}^{(k)}$, and $W_{1l}^{(k)}$ are functions of the same form as u_{14}, v_{14}, and w_{14} respectively. The complementary functions of these equations are the same as at each preceding step of the integration, and the complete solutions may also be obtained in the same way as equations (38) were determined. The equations analogous to (36) and (37) differ from (36) and (37) respectively only in the superscript k instead of 1. The complete solutions will therefore have the same form as (38), except that the respective particular integrals are functions similar to $P_1^{(k)}$, $Q_1^{(k)}$, and $R_1^{(k)}$ instead of $P_1^{(1)}$, $Q_1^{(1)}$, and $R_1^{(1)}$ as in (38). From condition (C_1) the arbitrary constants will all be zero, except $K_4^{(k)}$, and from condition (C_2) this constant is found to carry the factor $\varepsilon^k\alpha_1^{k+1}$. Hence the desired solutions for $p_1^{(k)}$, $q_1^{(k)}$, and $r_1^{(k)}$ are the same as (40) if $\nu = k$. This completes the induction.

When these solutions for $p_1^{(\nu)}$, $q_1^{(\nu)}$, and $r_1^{(\nu)}$, $(\nu=1, 2, \ldots\infty)$ are substituted in (34) and (33), we obtain as the solutions of (11),

$$\left.\begin{aligned}p_1&=\sum_{j=1}^{\infty}\sum_{k=1}^{j}e^{-k\rho_1\tau}u_{1k}^{(j)}(\tau)\varepsilon^{j-1}\alpha_1^j\gamma^j,\\ q_1&=\sum_{j=1}^{\infty}\sum_{k=1}^{j}e^{-k\rho_1\tau}v_{1k}^{(j)}(\tau)\varepsilon^{j-1}\alpha_1^j\gamma^j,\\ r_1&=\sum_{j=1}^{\infty}\sum_{k=1}^{j}e^{-k\rho_1\tau}w_{1k}^{(j)}(\tau)\varepsilon^{j-1}\alpha_1^j\gamma^j,\end{aligned}\right\}\tag{41}$$

where the superscript on the functions $u_{1k}^{(j)}(\tau)$, $v_{1k}^{(j)}(\tau)$, and $w_{1k}^{(j)}(\tau)$ has been made to conform with the corresponding power of γ. Since the two arbitrary parameters α_1 and γ occur only in products as indicated, we may suppress either without loss of generality. Let us suppose that $\alpha_1=1$.

Equations (41) are the asymptotic solutions of (11) which approach zero as τ and therefore t approach $+\infty$. In the same way there could be obtained the asymptotic solutions which approach zero as τ and therefore t approach $-\infty$. It is not necessary to construct these solutions, however, since they can be obtained directly from (41), as we proceed to show.

It was shown in the two paragraphs following equation (20) that if a set of solutions of equations (11) is known, then another set can be obtained by changing the signs of τ, q_1, and r_1 in the former set. On making these changes in (41) we thus obtain the asymptotic solutions of (11) which approach zero as τ approaches $-\infty$, viz.,

$$\left.\begin{aligned}p_1&=+\sum_{j=1}^{\infty}\sum_{k=1}^{j}e^{k\rho_1\tau}u_{1k}^{(j)}(-\tau)\varepsilon^{j-1}\gamma^j,\\ q_1&=-\sum_{j=1}^{\infty}\sum_{k=1}^{j}e^{k\rho_1\tau}v_{1k}^{(j)}(-\tau)\varepsilon^{j-1}\gamma^j,\\ r_1&=-\sum_{j=1}^{\infty}\sum_{k=1}^{j}e^{k\rho_1\tau}w_{1k}^{(j)}(-\tau)\varepsilon^{j-1}\gamma^j.\end{aligned}\right\}\tag{42}$$

Upon substituting (41) and (42) in (10), and returning to the original variables ξ, η, ζ through the substitutions (4) and (2), the asymptotic solutions of the original differential equations (1) in terms of τ are found to be

$$\left.\begin{aligned}\xi&=\xi_0+\bar{x}_1+\sum_{j=1}^{\infty}\sum_{k=1}^{j}e^{\mp k\rho_1\tau}u_{1k}^{(j)}(\pm\tau)\varepsilon^j\gamma^j,\\ \eta&=0+\bar{y}_1\pm\sum_{j=1}^{\infty}\sum_{k=1}^{j}e^{\mp k\rho_1\tau}v_{1k}^{(j)}(\pm\tau)\varepsilon^j\gamma^j,\\ \zeta&=0+\bar{z}_1\pm\sum_{j=1}^{\infty}\sum_{k=1}^{j}e^{\mp k\rho_1\tau}w_{1k}^{(j)}(\pm\tau)\varepsilon^j\gamma^j.\end{aligned}\right\}\tag{43}$$

Where the double signs occur, the upper signs give the solutions which approach the periodic orbits as τ approaches $+\infty$, and the lower signs give the solutions which approach the same periodic orbits as τ approaches $-\infty$.

Asymptotic Orbits of Class B.

We consider now the orbits which are asymptotic to the periodic orbits of Class B. The method of constructing these orbits is entirely similar to the one used in constructing the asymptotic orbits of Class A.

On putting $j=2$ and considering the differential equations obtained by equating the coefficients of the various powers of γ in (11′), that is, in (11) when (33) and (34) have been substituted and γ divided out, we obtain solutions which will give the asymptotic orbits, provided the arbitrary constants of integration are so chosen that conditions (C_1) and (C_2) are satisfied.

The differential equations arising from the terms independent of γ in (11′) are the same as (12B), except for the superscript 0 on p_2, q_2, and r_2. Their solutions are therefore the same as (31) and (32). When the conditions (C_1) and (C_2) are imposed, we obtain

$$p_2^{(0)}=\alpha_2 e^{-\rho_2\tau}u_{22}, \quad q_2^{(0)}=-\alpha_2 e^{-\rho_2\tau}v_{22}, \quad r_2^{(0)}=0.$$

From the coefficients of γ to the first degree in (11′) we obtain the differential equations which define $p_2^{(1)}$, $q_2^{(1)}$, and $r_2^{(1)}$. These equations have the same left members as (12B), except the superscript 1. Let the respective right members be denoted by $P_2^{(1)}$, $Q_2^{(1)}$, and $R_2^{(1)}$. Then

$$P_2^{(1)}=\varepsilon\alpha_2^2 e^{-2\rho_2\tau}U_{22}^{(1)}, \quad Q_2^{(1)}=\varepsilon\alpha_2^2 e^{-2\rho_2\tau}V_{22}^{(1)}, \quad R_2^{(1)}=0,$$

where $U_{22}^{(1)}$ and $V_{22}^{(1)}$ are functions of the same form as u_{22} and v_{22} respectively. The complementary functions of the differential equations are the same as (31) and (32), but let the arbitrary constants be denoted by $l_1^{(1)}, \ldots, l_6^{(1)}$. By using the method of the variation of parameters to obtain the complete solutions we have

$$\left.\begin{aligned}
&\dot{l}_1^{(1)}e^{\rho_2\tau}u_{21}+\dot{l}_2^{(1)}e^{-\rho_2\tau}u_{22}+\dot{l}_3^{(1)}u_{23}+\dot{l}_4^{(1)}(u_{24}+L\tau u_{23})=0,\\
&\dot{l}_1^{(1)}e^{\rho_2\tau}(\rho_2u_{21}+\dot{u}_{21})+\dot{l}_2^{(1)}e^{-\rho_2\tau}(-\rho_2u_{22}+\dot{u}_{22})+\dot{l}_3^{(1)}\dot{u}_{23}\\
&\qquad\qquad\qquad+\dot{l}_4^{(1)}[\dot{u}_{24}+L(u_{23}+\tau\dot{u}_{23})]=P_2^{(1)},\\
&\dot{l}_1^{(1)}e^{\rho_2\tau}v_{21}-\dot{l}_2^{(1)}e^{-\rho_2\tau}v_{22}+\dot{l}_3^{(1)}v_{23}+\dot{l}_4^{(1)}(v_{24}+L\tau v_{23})=0,\\
&\dot{l}_1^{(1)}e^{\rho_2\tau}(\rho_2v_{21}+\dot{v}_{21})+\dot{l}_2^{(1)}e^{-\rho_2\tau}(\rho_2v_{22}-\dot{v}_{22})+\dot{l}_3^{(1)}\dot{v}_{23}\\
&\qquad\qquad\qquad+\dot{l}_4^{(1)}[\dot{v}_{24}+L(v_{23}+\tau\dot{v}_{23})]=Q_2^{(1)},\\
&\dot{l}_5^{(1)}e^{i\omega\tau}w_{21}+\dot{l}_6^{(1)}e^{-i\omega\tau}w_{22}=0,\\
&\dot{l}_5^{(1)}e^{i\omega\tau}(i\omega w_{21}+\dot{w}_{21})-\dot{l}_6^{(1)}e^{-i\omega\tau}(i\omega w_{22}-\dot{w}_{22})=0.
\end{aligned}\right\} \quad (44)$$

The last two equations are independent of the first four. Since the determinant of the coefficients of $\dot{l}_5^{(1)}$ and $\dot{l}_6^{(1)}$ in these last two equations is different from zero, viz., Δ_3, the only solutions for $\dot{l}_5^{(1)}$ and $\dot{l}_6^{(1)}$ are $\dot{l}_5^{(1)}=\dot{l}_6^{(1)}=0$.

The determinant of the coefficients of $\dot{l}_1^{(1)}, \ldots, \dot{l}_4^{(1)}$ is different from zero, viz., Δ_2 equation (30), and therefore the first four equations of (44) can be solved for $\dot{l}_1^{(1)}, \ldots, \dot{l}_4^{(1)}$. The resulting solutions are

$$\dot{l}_k^{(1)} = \frac{\Delta_{2k}^{(1)}}{\Delta_2}, \qquad (k=1, \ldots, 4), \tag{45}$$

where $\Delta_{2k}^{(1)}$ is the determinant obtained by replacing the elements of the k-th row of Δ_2 with 0, $P_2^{(1)}$, 0, and $Q_2^{(1)}$ respectively. Since the right members $P_2^{(1)}$ and $Q_2^{(1)}$ do not contain terms in $e^{\pm\rho_2\tau}$, the integrations of (45) for $l_1^{(1)}$ and $l_2^{(1)}$ will not contain terms in τ explicitly. Such terms arise, however, in the integration for $l_3^{(1)}$ and $l_4^{(1)}$, but when they are substituted in the complementary functions they cancel off. The complete solutions of the differential equations at this step are thus found to be

$$\left.\begin{aligned} p_2^{(1)} &= L_1^{(1)} e^{\rho_2\tau} u_{21} + L_2^{(1)} e^{-\rho_2\tau} u_{22} + L_3^{(1)} u_{23} + L_4^{(1)} (u_{24} + L\tau u_{23}) + \varepsilon\alpha_2^2 e^{-2\rho_2\tau} u_{23}^{(1)}(\tau), \\ q_2^{(1)} &= L_1^{(1)} e^{\rho_2\tau} v_{21} - L_2^{(1)} e^{-\rho_2\tau} v_{22} + L_3^{(1)} v_{23} + L_4^{(1)} (v_{24} + L\tau v_{23}) + \varepsilon\alpha_2^2 e^{-2\rho_2\tau} v_{23}^{(1)}(\tau), \\ r_2^{(1)} &= L_5^{(1)} e^{i\omega\tau} w_{21} + L_6^{(1)} e^{-i\omega\tau} w_{22}, \end{aligned}\right\} \tag{46}$$

where $L_1^{(1)} \ldots, L_6^{(1)}$ are the constants of integration, and $u_{23}^{(1)}(\tau)$ and $v_{23}^{(1)}(\tau)$ are functions similar to u_{21} and v_{21} respectively. From condition (C_1) we have

$$L_1^{(1)} = L_3^{(1)} = L_4^{(1)} = L_5^{(1)} = L_6^{(1)} = 0,$$

and from condition (C_2)

$$L_2^{(1)} = -\varepsilon\alpha_2^2 u_{23}^{(1)}(0).$$

Then the solutions (46) become

$$\begin{aligned} p_2^{(1)} &= \varepsilon\alpha_2^2 [e^{-\rho_2\tau} u_{21}^{(1)}(\tau) + e^{-2\rho_2\tau} u_{22}^{(1)}(\tau)], \\ q_2^{(1)} &= \varepsilon\alpha_2^2 [e^{-\rho_2\tau} v_{21}^{(1)}(\tau) + e^{-2\rho_2\tau} v_{22}^{(1)}(\tau)], \\ r_2^{(1)} &= 0, \end{aligned}$$

where $u_{2k}^{(1)}(\tau)$ and $v_{2k}^{(1)}(\tau)$, $(k=1, 2)$, are similar to u_{21} and v_{21} respectively.

The remaining steps of the integration can be carried on in the same way. Proceeding by induction to the general term, we find that

$$p_2^{(\nu)} = \varepsilon^\nu \alpha_2^{\nu+1} \sum_{k=1}^{\nu+1} e^{-k\rho_2\tau} u_{2k}^{(\nu)}(\tau), \quad q_2^{(\nu)} = \varepsilon^\nu \alpha_2^{\nu+1} \sum_{k=1}^{\nu+1} e^{-k\rho_2\tau} v_{2k}^{(\nu)}(\tau), \quad r_2^{(\nu)} = 0,$$

where the functions $u_{2k}^{(\nu)}(\tau)$ and $v_{2k}^{(\nu)}(\tau)$ have the same form as u_{21} and v_{21} respectively. When these terms are substituted in (34) and (33), we find the solutions of (11) to be

$$p_2 = \sum_{j=1}^{\infty} \sum_{k=1}^{j} e^{-k\rho_2\tau} u_{2k}^{(j)}(\tau) \varepsilon^{j-1} \alpha_2^j \gamma^j, \quad q_2 = \sum_{j=1}^{\infty} \sum_{k=1}^{j} e^{-k\rho_2\tau} v_{2k}^{(j)}(\tau) \varepsilon^{j-1} \alpha_2^j \gamma^j, \quad r_2 = 0. \tag{47}$$

As in the solutions (41) we may put $\alpha_2 = 1$. These equations (47) are the

asymptotic solutions which approach zero as τ approaches $+\infty$. By changing the signs of τ and q_2 as in equations (42), we obtain

$$p_2=+\sum_{j=1}^{\infty}\sum_{k=1}^{j}e^{k\rho_2\tau}u_{2k}^{(j)}(-\tau)\varepsilon^{j-1}\gamma^j,\quad q_2=-\sum_{j=1}^{\infty}\sum_{k=1}^{j}e^{k\rho_2\tau}v_{2k}^{(j)}(-\tau)\varepsilon^{j-1}\gamma^j,\quad r_2=0,\quad (48)$$

which are the asymptotic solutions approaching zero as τ approaches $-\infty$. Since $r_2=0$ in (47) and (48), the asymptotic orbits are of two dimensions and are complanar with the periodic orbits of Class B.

In terms of the original variables ξ, η, ζ, these asymptotic solutions of equations (1) are

$$\left.\begin{aligned}\xi&=\xi_0+\overline{x}_2+\sum_{j=1}^{\infty}\sum_{k=1}^{j}e^{\mp k\rho_2\tau}u_{2k}^{(j)}(\pm\tau)\varepsilon^j\gamma^j,\\ \eta&=0+\overline{y}_2\pm\sum_{j=1}^{\infty}\sum_{k=1}^{j}e^{\mp k\rho_2\tau}v_{2k}^{(j)}(\pm\tau)\varepsilon^j\gamma^j,\\ \zeta&=0.\end{aligned}\right\}\quad (49)$$

These solutions approach the periodic orbits of Class B as τ approaches $+\infty$ or $-\infty$ according respectively as the upper or lower signs are taken.

§ 9. Geometrical Considerations.

The asymptotic solutions (43) and (49) contain the three undetermined constants t_0, ε, and γ. The constant t_0 denotes the initial time and may be put equal to zero without loss of generality. The parameters ε and γ are the respective scale factors of the periodic and asymptotic orbits. The physical interpretation of the parameter ε has already been referred to in the latter part of § 4. From the way in which the initial conditions were chosen in (C_2), it is evident that $\varepsilon\gamma$ denotes the ξ-component of the infinitesimal body's initial displacement from the periodic orbit.

Let us now consider the directions in which the asymptotic orbits approach the periodic orbits. We shall discuss only the orbits which approach the periodic orbits of Class A as τ approaches $+\infty$, since the discussion is entirely similar for the other asymptotic orbits.

As τ becomes very large, the most important terms of the solutions of (11) are those in $e^{-\rho_1\tau}$. These terms arise only from the complementary functions of the differential equations which define the various $p_1^{(k)}$, $q_1^{(k)}$, and $r_1^{(k)}$ in (34). Neglecting the explicit values of the constants of integration which are associated with $e^{-\rho_1\tau}$ at each stage of the integration, we find that the predominating terms of the solutions as τ approaches $+\infty$ are

$$\begin{aligned}p_1&=+e^{-\rho_1\tau}u_{14}[K_4^{(0)}\gamma+K_4^{(1)}\gamma^2+K_4^{(2)}\gamma^3+\ldots.],\\ q_1&=-e^{-\rho_1\tau}v_{14}[K_4^{(0)}\gamma+K_4^{(1)}\gamma^2+K_4^{(2)}\gamma^3+\ldots.],\\ r_1&=-e^{-\rho_1\tau}w_{14}[K_4^{(0)}\gamma+K_4^{(1)}\gamma^2+K_4^{(2)}\gamma^3+\ldots.],\end{aligned}$$

where $K_4^{(0)}$, $K_4^{(1)}$, ..., are the constants of integration.

The projection on the $\xi\eta$-plane of the direction of the approach is

$$\lim_{\tau=+\infty} \frac{dq_1}{d\tau}\Big/\frac{dp_1}{d\tau} = \lim_{\tau=+\infty} \frac{\dot{v}_{14}-\rho_1 v_{14}}{\rho_1 u_{14}-\dot{u}_{14}}.$$

This limit is independent of γ, but, since u_{14} and v_{14} contain sines and cosines of multiples of $\sqrt{A}\tau$, it is indeterminate. The projections on the $\xi\zeta$- and $\eta\zeta$-planes are likewise found to be indeterminate. Similar results are obtained for the orbits which approach the periodic orbits of Class A as τ approaches $-\infty$, and also for the corresponding orbits of Class B.

§ 10. ILLUSTRATIVE EXAMPLES.

We shall conclude this article with numerical examples and diagrams of the periodic and asymptotic orbits which have been discussed in the preceding sections. In these examples the ratio of the finite masses is ten to one, or $1-\mu=10/11$ and $\mu=1/11$, being the ratio used in the particular periodic orbits already mentioned in § 4.

The solutions for the asymptotic orbits have been carried out to the third degree in ε for orbits of Class A and to the second degree in ε for orbits of Class B, but only to the first degree in γ for both classes of orbits.

In the numerical results that are to be found in the tables which follow, ε has the value 0.5 for Class A and 0.01 for Class B while $\gamma=0.1$ for both classes. The values of εx_j, εy_j, and εz_j $(j=1, 2)$ in the tables are the coordinates for the periodic orbits for the various values of τ indicated. The values for $\varepsilon\gamma p_j$, $\varepsilon\gamma q_j$, and $\varepsilon\gamma r_j$ are the amounts which must be added to the coordinates for the periodic orbits in order to obtain the asymptotic orbits. For $j=2$, $z_j=r_j=0$.

The periodic orbits are represented by the heavy lines in the diagrams, and the asymptotic orbits by the dotted lines. The arrows indicate the direction of motion. No arrows appear in the periodic orbits in Figs. 2, 5, and 8, as in these projections the infinitesimal body oscillates up and down along the same curve. The origin of coordinates is taken at the equilibrium point marked in the diagram, and the axes are parallel to the rotating $\xi\eta\zeta$ axes (see equations (2)). The unit of measurement is indicated in each diagram.

The following results have been obtained.

Orbits of Class A.

Equilibrium point (a).

$$\varepsilon p_1 = e^{-\rho_1\tau}\gamma[\varepsilon+\varepsilon^3(-0.29+0.29\cos 2\sqrt{A}\tau-1.08\sin 2\sqrt{A}\tau)+\ldots],$$
$$\varepsilon q_1 = e^{-\rho_1\tau}\gamma[-0.75\,\varepsilon+\varepsilon^3(0.22-0.61\cos 2\sqrt{A}\tau-0.56\sin 2\sqrt{A}\tau)+\ldots],$$
$$\varepsilon r_1 = e^{-\rho_1\tau}\gamma[\varepsilon^2(1.58\cos\sqrt{A}\tau+0.91\sin\sqrt{A}\tau)+\ldots],$$
$$\rho_1 = 1.83-0.23\,\varepsilon^2+\ldots,\quad A=2.548.$$

TABLE 3.

$\varepsilon=0.5,\ \gamma=0.1.$

τ	ϵx_1	ϵy_1	ϵz_1	$\epsilon\gamma p_1$	$\epsilon\gamma q_1$	$\epsilon\gamma r_1$
0	—.041	0	0	.0500	—.0420	.0400
.1	—.043	—.008	.043	.0385	—.0369	.0359
.2	—.047	—.015	.085	.0295	—.0309	.0315
.3	—.054	—.021	.125	.0223	—.0259	.0270
.4	—.064	—.026	.163	.0173	—.0212	.0227
.5	—.074	—.028	.198	.0136	—.0173	.0187
.6	—.084	—.028	.228	.0111	—.0142	.0155
.7	—.095	—.025	.254	.0093	—.0107	.0118
.8	—.104	—.021	.274	.0080	—.0083	.0091
.9	—.111	—.015	.289	.0071	—.0064	.0067
1.0	—.116	—.007	.297	.0065	—.0048	.0048
1.2	—.115	.009	.296	.0056	—.0029	.0020
1.4	—.103	.022	.271	.0045	—.0020	.0003
1.6	—.083	.028	.223	.0034	—.0015	—.0005 6
1.8	—.062	.028	.157	.0025	—.0013	—.0009 1
2.0	—.046	.014	.078	.0017	—.0011	—.0009 5
2.4	—.048	—.016	—.092	.0005 9	—.0006	—.0006 5
2.8	—.087	—.027	—.233	.0002 3	—.0002 7	—.0003 0
3.2	—.116	—.006	—.298	.0001 4	—.0000 9	—.0000 9
3.6	—.102	.022	—.269	.0000 92	—.0000 7	—.0000 4
4.0	—.060	.024	—.151	.0000 48	—.0000 3	—.0000 18
4.4	—.041	—.003	.014	.0000 15	—.0000 2	—.0000 16

The projections of the above orbits on the coordinate planes are given in Figs. 1, 2, and 3.

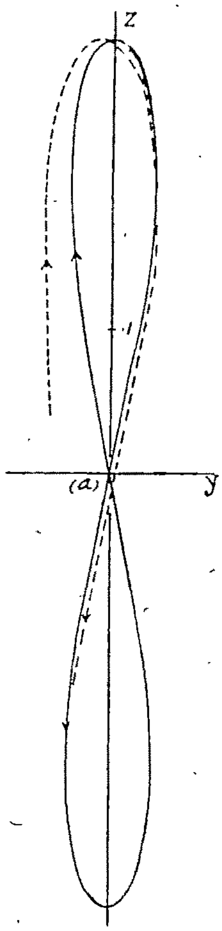


FIG. 1.

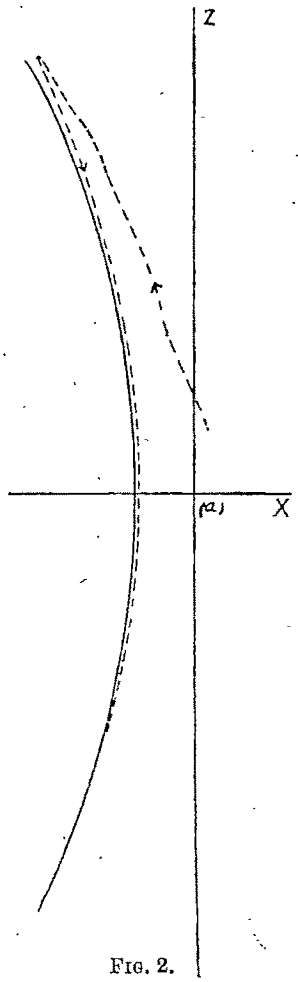


FIG. 2.

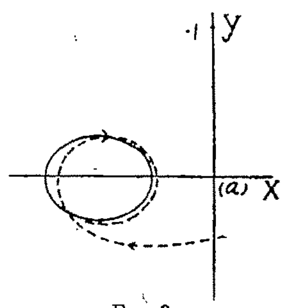


FIG. 3.

Equilibrium point (*b*).

$$\epsilon p_1 = e^{-\rho_1\tau}\gamma[\epsilon + \epsilon^3(0.222 - 0.222\cos 2\sqrt{A}\tau - 0.454 \sin 2\sqrt{A}\tau) + \ldots\ldots],$$
$$\epsilon q_1 = e^{-\rho_1\tau}\gamma[0.40\,\epsilon + \epsilon^2(0.089 - 0.182\cos 2\sqrt{A}\tau - 0.037 \sin 2\sqrt{A}\tau) + \ldots\ldots],$$
$$\epsilon r_1 = e^{-\rho_1\tau}\gamma[\epsilon(-0.345\sin\sqrt{A}\tau - 0.523\cos\sqrt{A}\tau) + \ldots\ldots],$$
$$\rho_1 = 3.366 - 2.837\,\epsilon^2 + \ldots\ldots, \quad A = 6.510.$$

TABLE 4.

$\epsilon = 0.5, \quad \gamma = 0.1.$

τ	ϵx_1	ϵy_1	ϵz_1	$\epsilon\gamma p_1$	$\epsilon\gamma q_1$	$\epsilon\gamma r_1$
0	.0135	0	0	.0500	.0188	—.0131
.1	.0147	.0022	.0474	.0300	.0119	—.0094 0
.2	.0178	.0038	.0917	.0186	.0073 8	—.0062 6
.3	.0221	.0045	.1355	.0118	.0051 7	—.0038 6
.4	.0266	.0040	.1602	.0077 9	.0034 5	—.0022 6
.5	.0300	.0025	.1799	.0051 9	.0022 7	—.0009 57
.6	.0315	.0004	.1878	.0034 9	.0014 7	—.0005 76
.7	.0307	—.0019	.1837	.0023 1	.0009 36	—.0002 24
.8	.0278	—.0036	.1677	.0014 9	.0005 74	—.0000 454
.9	.0236	—.0045	.1594	.0009 34	.0003 47	+.0000 350
1.0	.0206	—.0044	.1047	.0005 83	.0002 13	+.0000 616
1.2	.0136	—.0007	.0150	.0002 04	.0000 762	+.0000 497
1.4	.0166	.0034	—.0784	.0000 750	.0000 309	+.0000 363
1.6	.0252	.0043	—.1519	.0000 308	.0000 136	+.0000 0936
1.8	.0312	.0011	—.1867	.0000 137	.0000 0590	+.0000 0257
2.0	.0289	—.0032	—.1837	.0000 0594	.0000 0232	+.0000 0056 6
2.2	.0205	—.0044	—.1169	.0000 0233	.0000 0081 0	—.0000 0021 6
2.4	.0140	—.0014	—.0301	.0000 0086 6	.0000 0030 8	—.0000 0002 2
2.6	.0156	.0029	.0645	.0000 0029 9	.0000 0012 4	—.0000 0001 9

The projections of the above orbits on the coordinate planes are given in Figs. 4, 5, and 6.

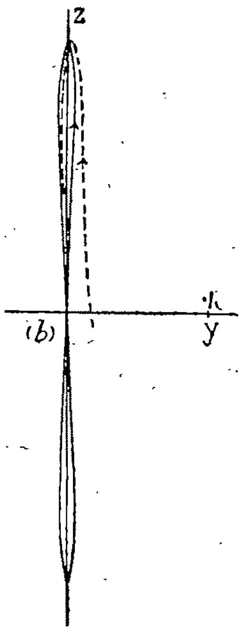


FIG. 4.

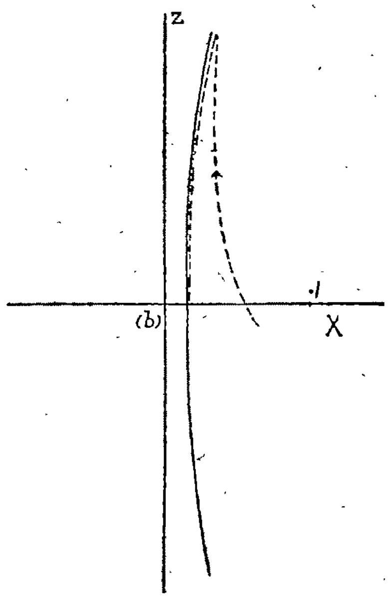


FIG. 5.

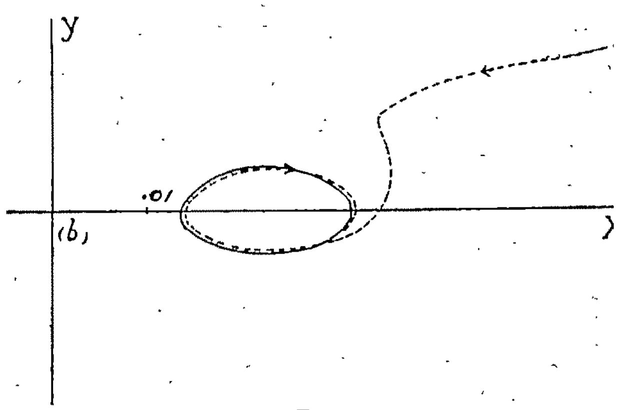


FIG. 6

Equilibrium point (*c*).

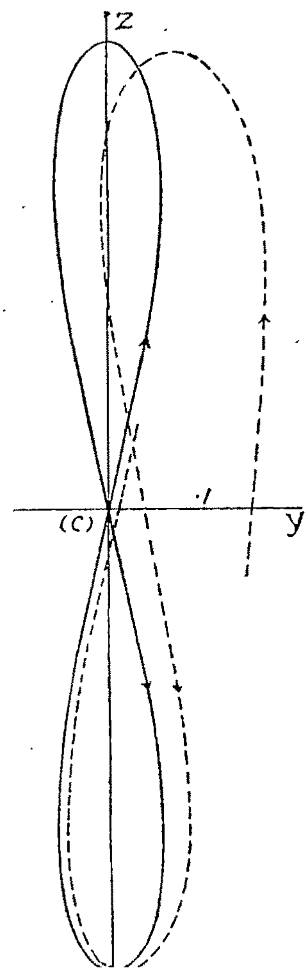
$$\epsilon p_1 = e^{-\rho_1\tau}\gamma[\epsilon + \epsilon^3(-0.104 + 0.104\cos 2\sqrt{A}\tau - 0.817 \sin 2\sqrt{A}\tau) + \ldots\ldots],$$
$$\epsilon q_1 = e^{-\rho_1\tau}\gamma[3.08\,\epsilon + \epsilon^3(-0.320 - 0.787\cos 2\sqrt{A}\tau - 0.106 \sin 2\sqrt{A}\tau) + \ldots\ldots],$$
$$\epsilon r_1 = e^{-\rho_1\tau}\gamma[\epsilon(-0.719\sin\sqrt{A}\tau - 3.145\cos\sqrt{A}\tau) + \ldots\ldots],$$
$$\rho_1 = 0.457 - 0.025\,\epsilon^2 + \ldots\ldots, \quad A = 1.082.$$

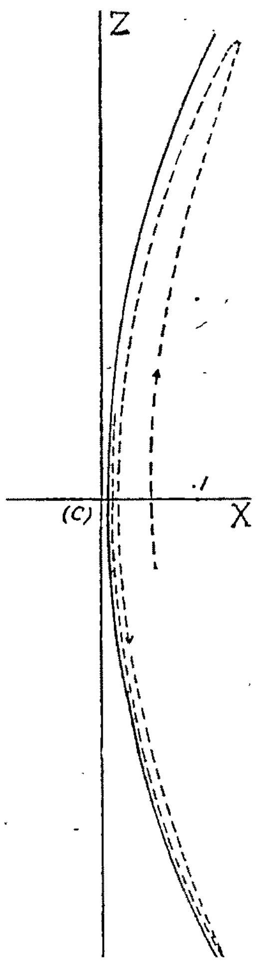
TABLE 5.

$\varepsilon=0.5, \quad \gamma=0.1.$

τ	ϵx_1	ϵy_1	ϵz_1	$\epsilon\gamma p_1$	$\epsilon\gamma q_1$	$\epsilon\gamma r_1$
0	.0050	0	0	.0500	.1402	—.0786
.2	.0099	.0228	.0989	.0418	.1279	—.0733
.4	.0238	.0418	.1939	.0349	.1180	—.0657
.6	.0443	.0536	.2803	.0297	.1098	—.0560
.8	.0678	.0563	.3547	.0264	.1030	—.0455
1.0	.0905	.0493	.4138	.0245	.0960	—.0346
1.2	.1084	.0340	.4550	.0236	.0893	—.0239
1.4	.1185	.0129	.4766	.0234	.0825	—.0139
1.6	.1190	—.0105	.4781	.0233	.0754	—.0051
1.8	.1099	—.0320	.4584	.0229	.0680	.0026
2.0	.0927	—.0481	.4190	.0221	.0612	.0088
2.2	.0703	—.0560	.3619	.0209	.0543	.0135
2.4	.0466	—.0543	.2890	.0191	.0481	.0168
2.6	.0256	—.0434	.2035	.0169	.0426	.0187
2.8	.0110	—.0251	.1094	.0147	.0377	.0194
3.0	.0051	—.0025	.0105	.0123	.0341	.0191
3.4	.0220	.0400	—.1843	.0086	.0289	.0161
3.8	.0653	.0564	—.3475	.0064 5	.0251	.0113
4.2	.1069	.0359	—.4517	.0057 3	.0218	.0060
4.6	.1194	—.0080	—.4790	.0056 2	.0184	.0014
5	.1101	—.0318	—.4723	.0051 0	.0151	.0015
5.4	.0491	—.0549	—.2971	.0046 6	.0119	—.0040
5.8	.0121	—.0272	—.1195	.0035 9	.0093	—.0047
6.2	.0081	.0182	.0787	.0014 5	.0076	—.0044

The projections of the above orbits on the coordinate planes are given in Figs. 7, 8, and 9.





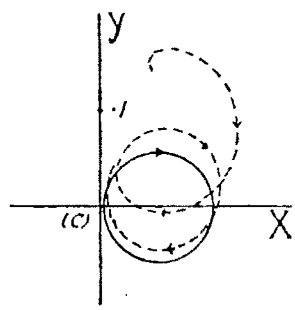


FIG. 9.

Equilibrium point (*a*).

$$\epsilon p_2 = e^{-\rho_2\tau}\gamma[\epsilon+\epsilon^2(-1.20+1.20\cos\sigma\tau+3.95\sin\sigma\tau)+\ldots],$$
$$\epsilon q_2 = e^{-\rho_2\tau}\gamma[0.75\,\epsilon-\epsilon^2(0.90+4.65\cos\sigma\tau+2.46\sin\sigma\tau)+\ldots],$$
$$\rho_2 = 1.833+(\ )\epsilon^2+\ldots,\quad \sigma=1.68.$$

TABLE 6.

$\epsilon=0.01, \quad \gamma=0.1.$

τ	ϵx_2	ϵy_2	$\epsilon\gamma p_2$	$\epsilon\gamma q_2$
0	.0100	0	.0010 00	.0007 13
.2	.0094	—.0088	.0007 00	.0004 77
.4	.0078	—.0166	.0004 90	.0003 31
.6	.0053	—.0225	.0003 42	.0002 31
.8	.0023	—.0259	.0002 38	.0001 63
1.0	—.0011	—.0264	.0001 63	.0001 16
1.2	—.0043	—.0240	.0001 13	.0000 820
1.4	—.0070	—.0189	.0000 776	.0000 581
1.6	—.0090	—.0117	.0000 534	.0000 412
1.8	—.0099	—.0031	.0000 361	.0000 289
2.0	—.0098	.0058	.0000 248	.0000 203
2.2	—.0085	.0140	.0000 169	.0000 140
2.4	—.0063	.0207	.0000 117	.0000 0970
2.6	—.0034	.0250	.0000 0809	.0000 0664
2.8	—.0001	.0266	.0000 0560	.0000 0452
3.0	.0032	.0252	.0000 0392	.0000 0306
3.2	.0062	.0210	.0000 0275	.0000 0208
3.4	.0084	.0144	.0000 0193	.0000 0141
3.6	.0097	.0062	.0000 0136	.0000 0095
3.8	.0100	.0027	.0000 0094	.0000 0065

These orbits are shown in Fig. 10.

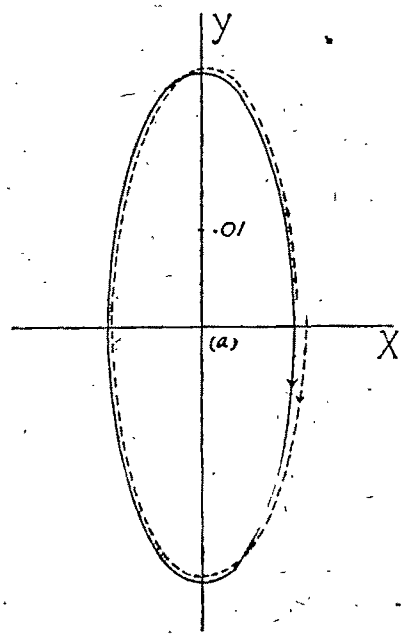


FIG. 10.

Equilibrium point (*b*).

$$\epsilon p_2 = e^{-\rho_2\tau}\gamma[\epsilon+\epsilon^2(0.69-0.69\cos\sigma\tau-2.47\sin\sigma\tau)+\ldots.],$$
$$\epsilon q_2 = e^{-\rho_2\tau}\gamma[0.40\,\epsilon+\epsilon^2(0.28+4.60\cos\sigma\tau+3.07\sin\sigma\tau)+\ldots.],$$
$$\rho_2 = 3.366+(\;)\epsilon^2+\ldots., \quad \sigma=2.61.$$

TABLE 7.

$\epsilon=0.01,\quad \gamma=0.1.$

τ	ϵx_2	ϵy_2	$\epsilon\gamma p_2$	$\epsilon\gamma q_2$
0	.0100	0	.0010 00	.0004 49
.1	.0097	—.0103	.0007 10	.0003 70
.2	.0087	—.0199	.0005 04	.0002 34
.3	.0071	—.0282	.0003 59	.0001 66
.4	.0050	—.0345	.0002 55	.0001 17
.5	.0026	—.0385	.0001 82	.0000 83
.6	.0000 4	—.0399	.0001 31	.0000 58
.7	—.0025	—.0386	.0000 933	.0000 398
.8	—.0050	—.0347	.0000 670	.0000 275
.9	—.0070	—.0284	.0000 481	.0000 190
1.0	—.0086	—.0202	.0000 345	.0000 131
1.2	—.0100	—.0003	.0000 178	.0000 063
1.4	—.0087	.0196	.0000 092	.0000 031
1.6	—.0051	.0344	.0000 047	.0000 016
1.8	—.0001	.0399	.0000 024	.0000 0087
2.0	+.0049	.0348	.0000 012	.0000 0047
2.2	+.0086	.0204	.0000 006	.0000 0026
2.4	.0100	.0006	.0000 003	.0000 0014

These orbits are shown in Fig. 11.

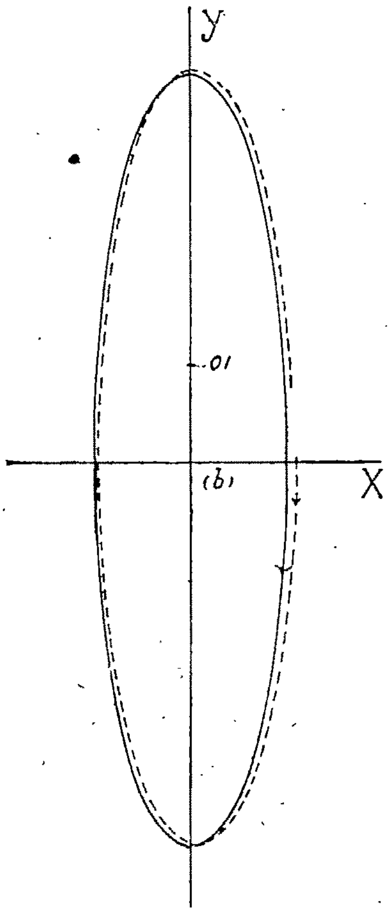


FIG. 11.

Equilibrium point (*c*).

$$\epsilon p_2 = e^{-\rho_2\tau}\gamma[\epsilon+\epsilon^2(25-25\cos\sigma\tau-23\sin\sigma\tau)+\ldots],$$
$$\epsilon q_2 = e^{-\rho_2\tau}\gamma[3.08\,\epsilon+\epsilon^2(77-56\cos\sigma\tau+31\sin\sigma\tau)+\ldots],$$
$$\rho_2 = 0.475+(\ )\epsilon^2+\ldots, \quad \sigma=1.07.$$

TABLE 8.

$\epsilon=0.01, \quad \gamma=0.1.$

τ	ϵx_2	ϵy_2	$\epsilon\gamma p_2$	$\epsilon\gamma q_2$
0	.0100	0	.0010 0	.0032 9
.1	.0099	—.0022	.0009 3	.0031 7
.2	.0098	—.0043	.0008 7	.0030 6
.3	.0095	—.0063	.0008 1	.0029 7
.4	.0091	—.0083	.0007 6	.0028 7
.5	.0086	—.0103	.0007 2	.0027 9
.6	.0080	—.0120	.0006 8	.0027 0
.7	.0073	—.0137	.0006 5	.0026 2
.8	.0066	—.0152	.0006 3	.0025 4
.9	.0057	—.0165	.0006 0	.0024 6
1.0	.0048	—.0176	.0005 8	.0023 9
1.5	—.0003	—.0199	.0005 4	.0020 5
2.0	—.0054	—.0169	.0004 6	.0017 1
2.5	—.0089	—.0090	.0004 2	.0013 7
3.0	—.0100	.0014	.0003 7	.0010 6
3.5	—.0082	.0140	.0003 0	.0007 8
4.0	—.0042	.0183	.0002 3	.0005 7
4.5	.0010	.0200	.0001 4	.0004 1
5.0	.0060	.0162	.0001 2	.0003 0
5.5	.0092	.0078	.0000 8	.0002 3
6.0	.0099	—.0027	.0000 6	.0001 9

These orbits are shown in Fig. 12.

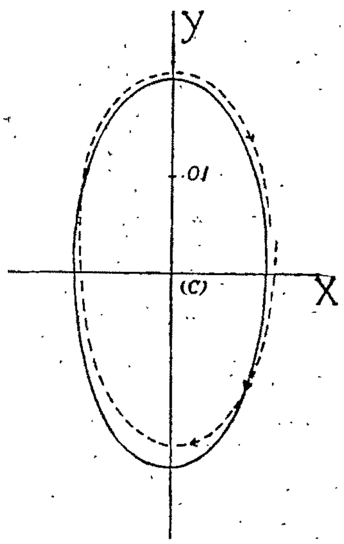


FIG. 12.

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Concerning the Invariant Theory of Involutions of Conics.

By Wayne Sensenig.

§ 1. Introduction.

The complete system of invariants of a single conic

$$f=a_x^2=a_{200}x_1^2+2a_{110}x_1x_2+\dots$$

consists of four members: $f=a_x^2$, $L=\alpha_u^2$, $D=a_\alpha^2$, and u_x. The complete system of invariants of two conics consists of twenty forms and this was derived first by Gordan and was first published in Clebsch's "Vorlesungen über Geometrie."* H. F. Baker derived the complete system for three, and incidentally two, conics.† Ciamberlini has published a determination of a complete system for three conics. ‡

The object of this paper is to derive and reduce in terms of the system of two conics a_x^2, b_x^2 the complete simultaneous system of the involution $K=a_x^2+kb_x^2$ and the harmonic conic $F=(\alpha\beta x)^2$. The complete system of K and F is expressed in terms of the complete system of the two base conics $f=a_x^2$ and $g=b_x^2$, which determine the configuration.

In Bôchers "Introduction to Higher Algebra," p. 164, it is shown that if two conics f, g intersect in four distinct points there exists a (non-singular) collineation which will reduce f and g simultaneously into the normal forms given below. Note that the equalities hold only by virtue of the equations of the collineation:

$$a_x^2=ax_1^2+bx_2^2+cx_3^2,\quad b_x^2=px_1^2+qx_2^2+rx_3^2. \tag{1}$$

In Table I, I give the well-known symbolical forms of the system of f and g, and also the normal system which I have computed from these symbolical forms by particularizing the coefficients of f and g, respectively, according to the following scheme:

$$a_{110}=a_{101}=a_{011}=0,\quad a_{200}=a,\quad a_{020}=b,\quad a_{002}=c,$$
$$b_{110}=b_{101}=b_{011}=0,\quad b_{200}=p,\quad b_{020}=q,\quad b_{002}=r.$$

Throughout the paper we have used the algorism for ternary transvection given by O. E. Glenn in the Transactions of the American Mathematical Society,

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* Cf. Osgood, American Journal of Mathematics, Vol. XIV (1892), p. 262, "System of two Simultaneous Ternary Quadratic Forms."

† *Cambridge Philosophical Transactions*, Vol. XV (1889–93), p. 62.

‡ *Giornale di Matematiche*, Vol. XXIV (1885–86), p. 141.

Vol. XVII (1916).* This algorism is analytical but it will suffice here to describe the transvectant τ of index $\binom{i\,j}{kl}$ of the two forms

$$\phi = Aa_{1x}a_{2x}\ldots a_{rx}\alpha_{1u}\ldots\alpha_{su}, \quad \psi = Bb_{1x}\ldots b_{px}\beta_{1u}\ldots\beta_{qu},$$

where A and B are constants, as AB/μ times the sum of all terms gotten by forming in the product $\phi\psi$, in all possible (μ) ways, i convolutions of the type $a_{1\beta_1}$, j of type (a_1b_1u), k of the type α_{1b_1}, and l of type $(\alpha_1\beta_1x)$. Then τ is abbreviated as

$$\tau = (\phi, \psi)^{ij}_{kl}.$$

In deriving the system of K and F we have evaluated each member as a transvectant and then reduced each form to a rational integral function of the members of the fundamental system of a_x^2 and b_x^2, expressing each one in the form of a polynomial in k. These reductions were performed by means of the following fundamental ternary identities, and their modifications:

$$\left.\begin{aligned} &(abc)d_x - (abd)c_x + (acd)b_x - (bcd)a_x \equiv 0, \\ &(abu)(\alpha\beta x) = \begin{vmatrix} a_\alpha & b_\alpha & u_\alpha \\ a_\beta & b_\beta & u_\beta \\ a_x & b_x & u_x \end{vmatrix}. \end{aligned}\right\} \tag{2}$$

The result for each case was then checked by direct verification, for the system of normal forms given in Table I. Table II contains the summary of the results of the reductions for all twenty concomitants in the system for K and F.

In Table III of the paper we give the fundamental system of the involution $a_x^2 + kb_x^2$ taken with a third general conic c_x^2. All concomitants in this system are expressed as rational integral functions of certain forms of the system for three conics as given by Baker; in fact in terms of the sixty-one forms of this system which are the simplest and therefore the most important in geometrical applications. This set of sixty-one concomitants is given below:

$f = a_x^2$, $g = b_x^2$, $\Phi_2 = (acu)^2$, $\Phi_1 = (bcu)^2$, $C_{12} = (acu)a_xc_x$, $C_{11} = (bcu)b_xc_x$,
$C_1 = a_xb_x(abu)$, $A_1 = a_\gamma^2$, $A_2 = b_\gamma^2$, $A_3 = c_\alpha^2$, $A_4 = c_\beta^2$, $C_{21} = a_\gamma a_x\gamma_u$,
$C_{22} = b_\gamma b_x\gamma_u$, $C_{23} = c_\alpha c_x\alpha_u$, $C_{24} = c_\beta c_x\beta_u$, $C_{42} = (\alpha\gamma x)\alpha_u\gamma_u$, $C_{41} = (\beta\gamma x)\beta_u\gamma_u$,
$F_2 = (\alpha\gamma x)^2$, $F_1 = (\beta\gamma x)^2$, $C_{51} = c_\alpha c_x\gamma_u(\alpha\gamma x)$, $C_{52} = c_\beta c_x\gamma_u(\beta\gamma x)$,
$C_{53} = a_\gamma a_x\alpha_u(\alpha\gamma x)$, $L'' = \gamma_u^2$, $C_{61} = a_\gamma c_x\gamma_u(acu)$, $C_{62} = b_\gamma c_x\gamma_u(bcu)$,
$C_{63} = a_xc_\alpha\alpha_u(acu)$, $C_{64} = b_xc_\beta\beta_u(bcu)$, $G_1 = a_\gamma c_\alpha a_xc_x(\alpha\gamma x)$, $G_2 = b_\gamma c_\beta b_xc_x(\beta\gamma x)$,
$\Gamma_2 = a_\gamma c_\alpha u_\alpha u_\gamma(acu)$, $\Gamma_1 = b_\gamma c_\beta u_\beta u_\gamma(bcu)$, $J = u_x$, $M = (abc)(abu)c_x$,
$N = (abc)^2$, $O_1 = a_\gamma b_x\gamma_u(abu)$, $O_2 = a_xb_\gamma\gamma_u(abu)$, $O_3 = b_xc_\alpha\alpha_u(bcu)$,
$O_4 = a_xc_\beta\beta_u(acu)$, $P = a_\gamma a_xb_\gamma b_x$, $Q_1 = a_\gamma b_xc_x\gamma_u(abc)$, $Q_2 = a_xb_\gamma c_x\gamma_u(abc)$,

* Cf. Glenn, "Theory of Invariants" (Boston, Ginn & Co., 1915), p. 227.

$R_1=(a'bc)(acu)(abu)a'_x$, $R_2=(ab'c)(bcu)(abu)b'_x$, $S_1=b_\alpha c_\alpha(bcu)$, $S_2=a_\beta c_\beta(acu)$, $S_3=a_\gamma b_\gamma(abu)$, $T_1=b_x b_\gamma \alpha_u(\alpha\gamma x)$, $T_2=b_x b_\alpha \gamma_u(\alpha\gamma x)$, $T_3=a_\gamma a_x\beta_u(\beta\gamma x)$, $T_4=a_\beta a_x u_\gamma(\beta\gamma x)$, $T_5=b_\gamma b_x\beta_u(\beta\gamma x)$, $U_1=b_\alpha b_\gamma(\alpha\gamma x)$, $U_2=a_\beta a_\gamma(\beta\gamma x)$, $V=a_x b_x c_x(abc)$, $W=a_\gamma b_\gamma c_x(abc)$, $X_1=b_\gamma c_\alpha b_x c_x(\alpha\gamma x)$, $X_2=a_\gamma c_\beta a_x c_x(\beta\gamma x)$, $Y=(acu)(abu)(bcu)$, $Z_1=b_\gamma c_\alpha u_\alpha u_\gamma(bcu)$, $Z_2=a_\gamma c_\beta u_\beta u_\gamma(acu)$; where those preceding J are concomitants of two conics and those following J, of three conics.

TABLE I.

Normal System of Two Conics.

1. $f\ =a_x^2=a_x'^2=a_x''^2=\text{etc.}=ax_1^2+bx_2^2+cx_3^2.$
2. $g\ =b_x^2=b_x'^2=b_x''^2=\text{etc.}=px_1^2+qx_2^2+rx_3^2.$
3. $D\ =a_\alpha^2=6abc.$
4. $D'\ =b_\beta^2=6pqr.$
5. $L\ =\alpha_u^2=2(bcu_1^2+acu_2^2+abu_3^2).$
6. $L'\ =\beta_u^2=2(qru_1^2+pru_2^2+pqu_3^2).$
7. $\Phi\ =(a_x^2,\ b_x^2)_{00}^{02}=(abu)^2=(br+cq)u_1^2+(ar+cp)u_2^2+(aq+bp)u_3^2.$
8. $C_1\ =(a_x^2,\ b_x^2)_{00}^{01}=(abu)a_xb_x$
   $=(aq-bp)x_1x_2u_3+(cp-ar)x_1x_3u_2+(br-cq)x_2x_3u_1.$
9. $A_{122}=(a_x^2,\ \beta_u^2)_{00}^{20}=a_\beta^2=2(aqr+bpr+cpq).$
10. $C_2\ =(a_x^2,\ \beta_u^2)_{00}^{10}=a_\beta a_x\beta_u=2(aqrx_1u_1+bprx_2u_2+cpqx_3u_3).$
11. $C_3\ =(\alpha_u^2,\ b_x^2)_{10}^{00}=\alpha_b\alpha_ub_x=2(bcpx_1u_1+acqx_2u_2+abrx_3u_3).$
12. $A_{112}=(\alpha_u^2,\ b_x^2)_{20}^{00}=\alpha_b^2=2(bcp+acq+abr).$
13. $C_4\ =(\alpha_u^2,\ \beta_u^2)_{01}^{00}=(\alpha\beta x)\alpha_u\beta_u$
    $=4[cr(bp-aq)u_1u_2x_3+bq(ar-cp)u_1u_3x_2+ap(cq-br)u_2u_3x_1].$
14. $F\ =(\alpha_u^2,\ \beta_u^2)_{02}^{00}=(\alpha\beta x)^2$
    $=4[ap(br+cq)x_1^2+bq(ar+cp)x_2^2+cr(bp+aq)x_3^2].$
15. $C_5\ =(\alpha_u^2,\ b_x^2\beta_u^2)_{11}^{00}=\alpha_b(\alpha\beta x)b_x\beta_u$
    $=4[aqr(br-cq)x_2x_3u_1+bpr(cp-ar)x_1x_3u_2+cpq(aq-bp)x_1x_2u_3].$
16. $C_6\ =(a_x^2,\ b_x^2\beta_u^2)_{00}^{11}=a_\beta(abu)b_x\beta_u$
    $=2[p^2(cq-br)u_2u_3x_1+q^2(ar-cp)u_1u_3x_2+r^2(bp-aq)u_1u_2x_3].$
17. $C_7\ =(a_x^2\alpha_u^2,\ b_x^2)_{10}^{01}=\alpha_b(abu)a_x\alpha_u$
    $=2[a^2(cq-br)u_2u_3x_1+b^2(ar-cp)u_1u_3x_2+c^2(bp-aq)u_1u_2x_3].$
18. $C_8\ =(a_x^2\alpha_u^2,\ \beta_u^2)_{01}^{10}=a_\beta(\alpha\beta x)a_x\alpha_u$
    $=4[abr(aq-bp)x_1x_2u_3+acq(cp-ar)x_1x_3u_2+bcp(br-cq)x_2x_3u_1].$
19. $G\ =(fL,\ gL')_{11}^{10}=a_\beta\alpha_b(\alpha\beta x)a_xb_x$
    $=4[a^2qr(br-cq)+b^2pr(cp-ar)+c^2pq(aq-bp)]x_1x_2x_3.$
20. $\Gamma\ =(fL,\ gL')_{10}^{11}=a_\beta\alpha_b(abu)\alpha_u\beta_u$
    $=4[p^2bc(cq-br)+q^2ac(ar-cp)+r^2ab(bp-aq)]u_1u_2u_3.$

§ 2. REDUCTION OF THE SYSTEM OF K AND F.

The complete system of K and F is obtained by forming and reducing the transvectants of K and F which correspond respectively to the transvectants of f, g required to produce the system of f, g (cf. Table I). Thus a member of the system of f, g is $\Phi=(f, g)^{02}_{00}$. Hence the corresponding form of the required system is

$$\Phi'=(K, F)^{02}_{00}.$$

I have not included in this paper all of the reductions which were necessary to produce Table II; but only typical ones. The most complicated cases of all were C_5', G', and Γ'. We add that the reductions for 3, 4, 5, 6 in Table II had been given previously.*

The Form Φ'.

$$\begin{aligned}
\Phi' &= (K, F)^{02}_{00} = (a_x^2+kb_x^2,\, (\alpha\beta)_x^2)^{02}_{00} = (a_x^2,\, (\alpha\beta)_x^2)^{02}_{00}+k(b_x^2,\, (\alpha\beta)_x^2)^{02}_{00}\\
&= (a\widehat{\alpha\beta}u)^2+k(b\widehat{\alpha\beta}u)^2 = (a_\alpha u_\beta-a_\beta u_\alpha)^2+k(b_\alpha u_\beta-b_\beta u_\alpha)^2\\
&= a_\alpha^2u_\beta^2+a_\beta^2u_\alpha^2-2a_\alpha a_\beta u_\alpha u_\beta+k(b_\alpha^2u_\beta^2+b_\beta^2u_\alpha^2-2b_\alpha b_\beta u_\alpha u_\beta)\\
&= DL'+A_{122}L-2(aa'a'')(a'a''u)a_\beta u_\beta\\
&\qquad +k[D'L+A_{112}L'-2(bb'b'')(b'b''u)b_\alpha u_\alpha]\\
&= DL'+A_{122}L-\tfrac{2}{3}DL'+k[D'L+A_{112}L'-\tfrac{2}{3}D'L]\\
&= \tfrac{1}{3}DL'+A_{122}L+k[\tfrac{1}{3}D'L+A_{112}L'].
\end{aligned}$$

The Form C_3'.

$$\begin{aligned}
C_3' &= (\alpha_u^2+2k(abu)^2+k^2\beta_u^2,\, (\gamma\delta)_x^2)^{00}_{10}, \text{ where } \alpha=\gamma \text{ and } \beta=\delta,\\
&= (\alpha_u^2,\, (\gamma\delta)_x^2)^{00}_{10}+2k[(ab)_u^2,\, (\gamma\delta)_x^2]^{00}_{10}+k^2[\beta_u^2,\, (\gamma\delta)_x^2]^{00}_{10}\\
&= (\alpha\gamma\delta)\alpha_u(\gamma\delta x)+2k(\widehat{ab}\gamma\delta)(abu)(\gamma\delta x)+k^2(\beta\gamma\delta)\beta_u(\gamma\delta x).
\end{aligned}$$

1. $$\begin{aligned}
(\alpha\gamma\delta)\alpha_u(\gamma\delta x) &= \tfrac{1}{2}(\alpha\gamma\delta)\{(\gamma\delta x)\alpha_u-(\alpha\delta x)\gamma_u\} = \tfrac{1}{2}(\alpha\gamma\delta)[(\alpha\gamma\delta)x_u-(\alpha\gamma x)\delta_u]\\
&= \tfrac{1}{2}(\alpha\gamma\delta)^2x_u-\tfrac{1}{2}(\alpha\gamma\delta)(\alpha\gamma x)\delta_u.
\end{aligned}$$

$$\begin{aligned}
(\alpha\gamma\delta)(\alpha\gamma x)\delta_u &= (\alpha\widehat{a'''a^{\mathrm{IV}}}\delta)(\alpha\widehat{a'''a^{\mathrm{IV}}}x)\delta_u = (a_\alpha'''a_\delta^{\mathrm{IV}}-a_\alpha^{\mathrm{IV}}a_\delta''')(a_\alpha'''a_x^{\mathrm{IV}}-a_\alpha^{\mathrm{IV}}a_x''')\delta_u\\
&= [(a'a''a''')(b'b''a^{\mathrm{IV}})-(a'a''a^{\mathrm{IV}})(b'b''a''')][(a'a''a''')a_x^{\mathrm{IV}}-(a'a''a^{\mathrm{IV}})a_x''']\delta_u\\
&= [(a'a''a''')^2a_\beta^{\mathrm{IV}}a_x^{\mathrm{IV}}-(a'a''a''')(a'a''a^{\mathrm{IV}})a_\beta^{\mathrm{IV}}a_x'''\\
&\qquad +(a'a''a^{\mathrm{IV}})^2a_\beta'''a_x'''-(a'a''a''')(a'a''a^{\mathrm{IV}})a_\beta'''a_x^{\mathrm{IV}}]\delta_u\\
&= D\cdot a_\beta^{\mathrm{IV}}a_x^{\mathrm{IV}}\beta_u-\tfrac{1}{3}D\cdot a_\beta^{\mathrm{IV}}a_x^{\mathrm{IV}}\beta_u+D\cdot a_\beta'''a_x'''\beta_u-\tfrac{1}{3}D\cdot a_\beta^{\mathrm{IV}}a_x^{\mathrm{IV}}\beta_u=\tfrac{4}{3}DC_2.
\end{aligned}$$

$\therefore$ $(\alpha\gamma\delta)(\gamma\delta x)\alpha_u=\tfrac{2}{3}DA_{122}u_x-\tfrac{2}{3}DC_2$; since $(\alpha\gamma\delta)^2=\tfrac{4}{3}DA_{122}$, $(\beta\gamma\delta)^2=\tfrac{4}{3}D'A_{112}$.

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* "Algebra of Invariants," by Grace and Young, p. 293, *Cambridge University Press*, 1903.

2. $(\widehat{ab\gamma\delta})(abu)(\gamma\delta x)=(a_\gamma b_\delta-a_\delta b_\gamma)(abu)(\alpha\beta x)=a_\alpha b_\beta(abu)(\alpha\beta x)$
$-a_\beta b_\alpha(abu)(\alpha\beta x)=a_\alpha b_\beta(abu)(\alpha\beta x)-k_8$
$=\frac{2}{9}DD'u_x-k_8$, since $a_\alpha b_\beta(abu)(\widehat{ab'b''}x)=a_\alpha b_\beta(abu)[b''_\alpha b'_x-b'_\alpha b''_x]$
$=a_\alpha b''_\alpha b_\beta b'_x(abu)-a_\alpha b'_\alpha b_\beta b''_x(abu)=(aa'a'')(a'a''b'')a_{(bu)}b_\beta b'_x$
$-(aa'a'')(a'a''b')a_{(bu)}b_\beta b''_x=\frac{1}{3}D(b''bu)(bb'b'')b'_x$
$-\frac{1}{3}D(b'bu)(bb'b'')b''_x=\frac{1}{3}D(bb''u)(bb''b')b'_x+\frac{1}{3}D(bb'u)(bb'b'')b''_x$
$=\frac{1}{9}DD'u_x+\frac{1}{9}DD'u_x=\frac{2}{9}DD'u_x.$

3. $(\beta\gamma\delta)(\gamma\delta x)\beta_u=\frac{1}{2}(\beta\gamma\delta)[(\gamma\delta x)\beta_u+(\beta\gamma x)\delta_u]=\frac{1}{2}(\beta\gamma\delta)[(\gamma\delta x)\beta_u+(\beta\gamma x)\delta_u]$
$=\frac{1}{2}(\beta\gamma\delta)[(\beta\gamma\delta)x_u+(\beta\delta x)\gamma_u]=\frac{1}{2}(\beta\gamma\delta)^2x_u+\frac{1}{2}(\beta\gamma\delta)(\beta\delta x)\gamma_u.$
$\frac{1}{2}(\beta\gamma\delta)(\beta\delta x)\gamma_u=\frac{1}{2}(\widehat{a'a''\beta'\beta})(\widehat{b'b''\beta'x})\alpha_u=\frac{1}{2}(a'_{\beta'}a''_\beta-a'_\beta a''_{\beta'})(b'_{\beta'}b''_x-b'_xb''_{\beta'})\alpha_u$
$=\frac{1}{2}[a'_{\beta'}a''_\beta b'_{\beta'}b''_x\alpha_u-a'_{\beta'}a''_\beta b'_xb''_{\beta'}\alpha_u-a'_\beta a''_{\beta'}b'_{\beta'}b''_x\alpha_u+a'_\beta a''_{\beta'}b'_xb''_{\beta'}\alpha_u]$
$=2a'_{\beta'}a''_\beta b'_{\beta'}b''_x\alpha_u=2(b'b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}a')b'_{(b''a'')}b''_x\alpha_u=\frac{2}{3}D'(a'b''a'')b''_x\alpha_u$
$=-\frac{2}{3}D'b''_\alpha b''_x\alpha_u=-\frac{2}{3}D'C_3.$

$\therefore(\beta\gamma\delta)(\gamma\delta x)\beta_u=\frac{2}{3}D'A_{112}u_x-\frac{2}{3}D'C_3.\quad k_8=a_\beta b_\alpha(abu)(\alpha\beta x)=\frac{1}{9}DD'u_x$
$+A_{112}C_2-A_{112}A_{122}u_x+A_{122}C_3.$

$\therefore C'_3=\frac{2}{3}DA_{122}u_x-\frac{2}{3}DC_2+2k\{\frac{1}{9}DD'u_x-A_{112}C_2+A_{112}A_{122}u_x-A_{122}C_3\}$
$+k^2(\frac{2}{3}D'A_{112}u_x-\frac{2}{3}D'C_3).$

The Form C'_7.

$C'_7=\{(a_x^2+kb_x^2)(\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2),(\alpha\beta)_x^2\}_{10}^{01}=\{a_x^2\alpha_u^2,(\alpha'\beta)_x^2\}_{10}^{01}$
$+2k\{a_x^2(a'''b)_u^2,(\alpha\beta)_x^2\}_{10}^{01}+k^2\{a_x^2\beta_u^2,(\alpha\beta')_x^2\}_{10}^{01}+k\{b_x^2\alpha_u^2,(\alpha'\beta)_x^2\}_{10}^{01}$
$+2k^2\{b_x^2(ab''')_u^2,(\alpha\beta)_x^2\}_{10}^{01}+k^3\{b_x^2\beta_u^2,(\alpha\beta')_x^2\}_{10}^{01}.$

1. $\{a_x^2\alpha_u^2,(\alpha'\beta)_x^2\}_{10}^{01}=(\alpha\alpha'\beta)(\widehat{a\alpha'\beta u})a_x\alpha_u=(\alpha\alpha'\beta)\{a_\beta\alpha'_u-a_{\alpha'}\beta_u\}a_x\alpha_u$
$=-(\alpha\alpha'\beta)a_{\alpha'}\beta_ua_x\alpha_u=-(aa'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}\widehat{\beta\alpha})a_x\alpha_u\beta_u$
$=-\frac{1}{3}D(\beta\alpha x)\alpha_u\beta_u=\frac{1}{3}DC_4.$

2. $\{a_x^2(a'''b)_u^2,(\alpha\beta)_x^2\}_{10}^{01}=(\widehat{a'''b\alpha\beta})(\widehat{a\alpha\beta u})a_x(a'''bu)$
$=(a'''_\alpha b_\beta-a'''_\beta b_\alpha)(a_\beta\alpha_u-a_\alpha\beta_u)a_x(a'''bu)=-a'''_\beta b_\alpha a_\beta\alpha_ua_x(a'''bu)$
$+a'''_\beta b_\alpha a_\alpha\beta_ua_x(a'''bu)\quad$ (Terms containing $a'''_\alpha b_\beta$ vanish.)
$=(-A_{122}C_7+\frac{1}{2}A_{112}C_4-\frac{1}{3}DC_6)+\frac{1}{3}DC_6=\frac{1}{2}A_{112}C_4-A_{122}C_7.$

For, $a'''_\beta a_\beta a_xb_\alpha\alpha_u(a'''bu)-a'''^2_\beta\cdot a_xb_\alpha\alpha_u(abu)$
$=a'''_\beta a_xb_\alpha\alpha_u\{(a'''bu)a_\beta-(abu)a'''_\beta\}=a'''_\beta a_xb_\alpha\alpha_u[(aa'''b)u_\beta-(aa'''u)b_\beta]$
$=\frac{1}{2}b_\alpha\alpha_uu_\beta(aa'''b)[a_xa'''_\beta-a_\beta a'''_x]-(bb'b'')(b'b''a''')b_\alpha a_x\alpha_u(aa'''u)$
$=\frac{1}{2}b_\alpha\alpha_uu_\beta\bar{\alpha}_b(\alpha x\beta)-\frac{1}{3}D'(a'''a'a'')(a'a''u)a'''_{(u\alpha)}a_x=\frac{1}{2}\alpha_b\alpha_u\alpha_b\bar{\alpha}_{(x\beta)}u_\beta.$

But, $\alpha_b\bar{\alpha}_u\bar{\alpha}_b\bar{\alpha}_{(x\beta)}u_\beta-\alpha_b^2\cdot\alpha_u u_\beta(\alpha x\beta)=\bar{\alpha}_b\alpha_u u_\beta[(\bar{\alpha}_x\beta)\alpha_b-(\alpha x\beta)\bar{\alpha}_b]$

$\equiv\bar{\alpha}_b\alpha_u u_\beta[(\alpha\bar{\alpha}x)b_\beta-(\alpha\bar{\alpha}\beta)b_x]\equiv(bb'b'')(b'b''u)\bar{\alpha}_b\alpha_u(\alpha\bar{\alpha}x)-\bar{\alpha}_b\alpha_u u_\beta(\alpha\bar{\alpha}\beta)b_x$

$=-\bar{\alpha}_b\alpha_u u_\beta b_x[a''_\beta a'_{\bar{\alpha}}-a'_\beta a''_{\bar{\alpha}}]=\bar{\alpha}_{a''}\bar{\alpha}_b a''_{(ua')}a'_\beta b_x u_\beta-\bar{\alpha}_{a'}\bar{\alpha}_b a'_{(a''u)}b_x u_\beta a''_\beta$

$=\frac{1}{3}D(bua')a'_\beta b_x u_\beta-\frac{1}{3}D(ba''u)b_x a''_\beta u_\beta=\frac{2}{3}D(abu)a_\beta b_x u_\beta=\frac{2}{3}DC_6.$

$\therefore a'''_\beta a_\beta a_x b_\alpha\alpha_u(a'''bu)=A_{122}C_7-\frac{1}{2}A_{112}C_4+\frac{1}{3}DC_6.$

3. $\{a_x^2\beta_u^2,\ (\alpha\beta')_x^2\}_{10}^{01}=(\beta\alpha\beta')(\widehat{a\alpha\beta'}u)a_x\beta_u=(\beta\alpha\beta')\{a_{\beta'}\alpha_u-a_\alpha\beta'_u\}a_x\beta_u$

$=a_{\beta'}\alpha_u a_x\beta_u\{b'_\alpha b''_{\beta'}-b'_{\beta'}b''_\alpha\}-\frac{1}{3}D(\beta'\beta x)\beta'_u\beta_u=(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}a)b''_{(ub')}\alpha_u a_x b'_\alpha$

$-(b'b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}a)b'_{(b''u)}\alpha_u a_x b''_\alpha-0=-\frac{2}{3}D'C_7.$

4. $\{b_x^2\alpha_u^2,(\alpha'\beta)_x^2\}_{10}^{01}=(\alpha\alpha'\beta)(\widehat{b\alpha'\beta}u)b_x\alpha_u=(\alpha\alpha'\beta)\{b_\beta\alpha'_u-\alpha'_b\beta_u\}b_x\alpha_u$

$=b_\beta\alpha'_u b_x\alpha_u(\alpha\alpha'\beta)-b_{\alpha'}\beta_u b_x\alpha_u(\alpha\alpha'\beta)=0-b_{\alpha'}\beta_u b_x\alpha_u\{a'_{\alpha'}a''_\beta-a'_\beta a''_{\alpha'}\}=\frac{2}{3}DC_6.$

5. $\{b_x^2(ab''')_u^2,(\alpha\beta)_x^2\}_{10}^{01}=(\widehat{ab'''}\alpha\beta)(b\widehat{\alpha\beta}u)b_x(ab'''u)$

$=(a_\alpha b'''_\beta-a_\beta b'''_\alpha)(b_\beta\alpha_u-b_\alpha\beta_u)b_x(ab'''u)=0-0-a_\beta b'''_\alpha b_\beta\alpha_u b_x(ab'''u)$

$+a_\beta b'''_\alpha b_\alpha\beta_u b_x(ab'''u)^*=-\frac{1}{3}D'C_7+[A_{112}C_6-\frac{1}{2}A_{122}C_4+\frac{1}{3}D'C_7\}$

$=A_{112}C_6-\frac{1}{2}A_{122}C_4$; * similar to 2.

6. $\{b_x^2\beta_u^2,(\alpha\beta')_x^2\}_{10}^{01}=(\beta\alpha\beta')(\widehat{b\alpha\beta'}u)b_x\beta_u=(b'_\alpha b''_{\beta'}-b'_{\beta'}b''_\alpha)(b_{\beta'}\alpha_u-b_\alpha\beta'_u)b_x\beta_u$

$=b'_\alpha b''_{\beta'}b_{\beta'}\alpha_u b_x\beta_u^{**}-b'_\alpha b''_{\beta'}b_\alpha\beta'_u b_x\beta_u-b'_{\beta'}b''_\alpha b_{\beta'}\alpha_u b_x\beta_u+b'_{\beta'}b''_\alpha b_\alpha\beta'_u b_x\beta_u.$

$^{**}=(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}b)b''_{(ub')}b'_\alpha b_x\alpha_u=\frac{1}{3}D'(bub')b'_\alpha b_x\alpha_u$

$=\frac{1}{6}D'(bub')\alpha_u[b'_\alpha b_x-b'_x b_\alpha]=\frac{1}{6}D'(b'bu)\alpha_u(\widehat{b'b}\alpha x)$

$=\frac{1}{6}D'\beta'''_u\alpha_u(\beta'''\alpha x)=-\frac{1}{6}D'C_4.$

Also, $b'_{\beta'}b''_\alpha b_{\beta'}\alpha_u b_x\beta_u=^{**}$ by $b'\sim b''$, and $-b'_\alpha b''_{\beta'}b_\alpha\beta'_u b_x\beta_u=b'_{\beta'}b''_\alpha b_\alpha\beta'_u b_x\beta_u=0.$

Hence $\{b_x^2\beta_u^2,\ (\alpha\beta')_x^2\}_{10}^{01}=-\frac{1}{3}D'C_4.$

$\therefore C'_7=\frac{1}{3}DC_4+k\{A_{112}C_4-2A_{122}C_7+\frac{2}{3}DC_6\}+k^2\{-A_{122}C_4+A_{112}C_6-\frac{2}{3}D'C_7\}$

$+k^3(-\frac{1}{3}D'C_4).$

The Form G'.

$G'=[(a_x^2+kb_x^2)(\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2),(\alpha\beta)_x^2(3a_\alpha^2a_\beta^2\beta_u^2+3b_\beta^2b_\alpha^2\alpha_u^2-2a_\alpha^2b_\beta^2(ab)_u^2]_{11}^{10}$

$=3a_\alpha^2a_\beta^2[(a_x^2\alpha_u^2,(\alpha'\beta)_x^2\beta_u'^2)_{11}^{10}+2k(a_x^2(a'''b)_u^2,(\alpha\beta)_x^2\beta_u'^2)_{11}^{10}+k^2(a_x^2\beta_u^2,(\alpha\beta')_x^2\beta_u''^2)_{11}^{10}$

$+k(b_x^2\alpha_u^2,(\alpha'\beta)_x^2\beta_u'^2)_{11}^{10}+2k^2(b_x^2(ab''')_u^2,(\alpha\beta)_x^2\beta_u'^2)_{11}^{10}+k^3(b_x^2\beta_u^2,(\alpha\beta')_x^2\beta_u''^2)_{11}^{10}]$

$+3b_\beta^2b_\alpha^2[(a_x^2\alpha_u^2,(\alpha'\beta)_x^2\alpha_u''^2)_{11}^{10}+2k(a_x^2(a'''b)_u^2,(\alpha\beta)_x^2\alpha_u'^2)_{11}^{10}$

$+k^2(a_x^2\beta_u^2,(\alpha\beta')_x^2\alpha_u'^2)_{11}^{10}+k(b_x^2\alpha_u^2,(\alpha'\beta)_x^2\alpha_u''^2)_{11}^{10}+2k^2(b_x^2(ab''')_u^2,(\alpha\beta)_x^2\alpha_u'^2)_{11}^{10}$

$+k^3(b_x^2\beta_u^2,(\alpha\beta')_x^2\alpha_u'^2)_{11}^{10}]-2a_\alpha^2b_\beta^2[(a_x^2\alpha_u^2,(\alpha'\beta)_x^2(a^{\mathrm{V}}b)_u^2)_{11}^{10}$

$+2k(a_x^2(a'''b)_u^2,(\alpha\beta)_x^2(a^{\mathrm{IV}}b''')_u^2)_{11}^{10}+k^2(a_x^2\beta_u^2,(\alpha\beta')_x^2(a'''b)_u^2)_{11}^{10}$

$+k(b_x^2\alpha_u^2,(\alpha'\beta)_x^2(ab''')_u^2)_{11}^{10}+2k^2(b_x^2(ab''')_u^2,(\alpha\beta)_x^2(a'''b^{\mathrm{IV}})_u^2)_{11}^{10}$

$+k^3(b_x^2\beta_u^2,(\alpha\beta')_x^2(ab^{\mathrm{V}})_u^2)_{11}^{10}].$

$$=3a_a^2a_\beta^2[a_{\beta'}(\alpha\alpha'\beta)(\alpha\beta'x)a_x(\alpha'\beta x)+2ka_{\beta'}(\widehat{a'''b}\alpha\beta)(\widehat{a'''b}\beta'x)a_x(\alpha\beta x)$$
$$+k^2a_{\beta''}(\beta\alpha\beta')(\beta\beta''x)a_x(\alpha\beta'x)+kb_{\beta'}(\alpha\alpha'\beta)(\alpha\beta'x)b_x(\alpha'\beta x)$$
$$+2k^2b_{\beta'}(\widehat{ab'''}\alpha\beta)(\widehat{ab'''}\beta'x)b_x(\alpha\beta x)+k^3b_{\beta''}(\beta\alpha\beta')(\beta\beta''x)b_x(\alpha\beta'x)]$$
$$+3b_\beta^2b_a^2[a_{\alpha''}(\alpha\alpha'\beta)(\alpha\alpha''x)a_x(\alpha'\beta x)+2ka_{\alpha'}(\widehat{a'''b}\alpha\beta)(\widehat{a'''b}\alpha'x)a_x(\alpha\beta x)$$
$$+k^2a_{\alpha'}(\beta\alpha\beta')(\beta\alpha'x)a_x(\alpha\beta'x)+kb_{\alpha''}(\alpha\alpha'\beta)(\alpha\alpha''x)b_x(\alpha'\beta x)$$
$$+2k^2b_{\alpha'}(\widehat{ab'''}\alpha\beta)(\widehat{ab'''}\alpha'x)b_x(\alpha\beta x)+k^3b_{\alpha'}(\beta\alpha\beta')(\beta\alpha'x)b_x(\alpha\beta'x)]$$
$$-2a_a^2b_\beta^2[(aa^{\mathrm{V}}b)(\alpha\alpha'\beta)(\alpha\widehat{a^{\mathrm{V}}b}x)a_x(\alpha'\beta x)$$
$$+2k(aa^{\mathrm{IV}}b''')(\widehat{a'''b}\alpha\beta)(\widehat{a'''b}\widehat{a^{\mathrm{IV}}b'''}x)a_x(\alpha\beta x)$$
$$+k^2(aa'''b)(\beta\alpha\beta')(\beta\widehat{a'''b}x)a_x(\alpha\beta'x)+k(bab''')(\alpha\alpha'\beta)(\alpha\widehat{ab'''}x)b_x(\alpha'\beta x)$$
$$+2k^2(ba'''b^{\mathrm{IV}})(\widehat{ab'''}\alpha\beta)(\widehat{ab'''}\widehat{a'''b^{\mathrm{IV}}}x)b_x(\alpha\beta x)$$
$$+k^3(bab^{\mathrm{V}})(\beta\alpha\beta')(\beta\widehat{ab^{\mathrm{V}}}x)b_x(\alpha\beta'x)].$$
$$=3D\,A_{122}[0+2k(-A_{122}G)+k^2(-\tfrac{2}{3}D'G)+k(0)+2k^2(0)+k^3(0)]$$
$$+3D'A_{112}[0+2k(0)+k^2(0)+k(\tfrac{2}{3}DG)+2k^2(A_{112}G)+k^3(0)]$$
$$-2DD'[-\tfrac{1}{3}DG+2k(-\tfrac{1}{2}A_{112}G)+k^2(0)+k(0)$$
$$+2k^2(\tfrac{1}{2}A_{122}G)+k^3(\tfrac{1}{3}D'G)]. \qquad (3)$$

Collecting terms,

$$G'=\{\tfrac{4}{3}D^2D'+k(4DD'A_{112}-6DA_{122}^2)+k^2(6D'A_{112}^2-4DD'A_{122})-\tfrac{4}{3}DD'^2\cdot k^3\}G.$$

The calculations are shown below:

1. $a_{\beta'}(\alpha\alpha'\beta)(\alpha\beta'x)a_x(\alpha'\beta x)=\tfrac{1}{2}a_{\beta'}a_x(\alpha\alpha'\beta)[(\alpha\beta'x)\alpha'_{(\beta x)}-(\alpha'\beta'x)\alpha_{(\beta x)}]$
$\equiv\tfrac{1}{2}a_{\beta'}a_x(\alpha\alpha'\beta)(\alpha\alpha'x)(\beta'\beta x)=\tfrac{1}{4}a_x(\alpha\alpha'x)(\beta\beta'x)[(\alpha\alpha'\beta)\beta'_a-(\alpha\alpha'\beta')\beta_a]$
$\equiv\tfrac{1}{4}a_x(\alpha\alpha'x)(\beta\beta'x)[(\alpha'\beta\beta')a_a-(\alpha\beta\beta')a_{\alpha'}]=0.$

2. $b_{\beta'}(\alpha\alpha'\beta)(\alpha\beta'x)b_x(\alpha'\beta x)=(bb'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b_x(\alpha\alpha'\beta)(\alpha'\beta x)$
$=\tfrac{1}{3}D'(\widehat{x\alpha x})$, etc. $=0$.

3. $a_{\alpha''}(\alpha\alpha'\beta)(\alpha\alpha''x)a_x(\alpha'\beta x)=(aa^{\mathrm{V}}a^{\mathrm{VI}})(a^{\mathrm{V}}a^{\mathrm{VI}}\widehat{x\alpha})a_x$, etc. $=0$.

4. $b_{\alpha''}(\alpha\alpha'\beta)(\alpha\alpha''x)b_x(\alpha'\beta x)=b_{\alpha''}(\alpha\alpha'\beta)[a'_{\alpha''}a''_x-a'_xa''_{\alpha''}]b_x(\alpha'\beta x)$
$=b_{\alpha''}(\alpha\alpha'\beta)a'_{\alpha''}a''_xb_x(\alpha'\beta x)-b_{\alpha''}(\alpha\alpha'\beta)a'_xa''_{\alpha''}b_x(\alpha'\beta x)$
$=(a'a^{\mathrm{V}}a^{\mathrm{VI}})(a^{\mathrm{V}}a^{\mathrm{VI}}b)[a'_{\alpha'}a''_\beta-a'_\beta a''_{\alpha'}]a''_xb_x(\alpha'\beta x)$
$-(a''a^{\mathrm{V}}a^{\mathrm{VI}})(a^{\mathrm{V}}a^{\mathrm{VI}}b)[a'_{\alpha'}a''_\beta-a'_\beta a''_{\alpha'}]a'_xb_x(\alpha'\beta x)=\tfrac{1}{3}Db_{\alpha'}a''_\beta a''_xb_x(\alpha'\beta x)$
$-\tfrac{1}{3}Db_\beta a''_{\alpha'}a''_xb_x(\alpha'\beta x)-\tfrac{1}{3}Db_\beta a'_{\alpha'}a'_xb_x(\alpha'\beta x)+\tfrac{1}{3}Db_{\alpha'}a'_\beta a'_xb_x(\alpha'\beta x)$
$=\tfrac{2}{3}Db_{\alpha'}a''_\beta a''_xb_x(\alpha'\beta x)-\tfrac{2}{3}D(bb'b'')(b'b''\widehat{x\alpha'})b_xa''_xa''_{\alpha'}=\tfrac{2}{3}DG-0.$

5. $(aa^{\mathrm{V}}b)(\alpha\alpha'\beta)(\widehat{\alpha a^{\mathrm{V}}bx})a_x(\alpha'\beta x)$

$$\begin{aligned}
&=\{a'_{\alpha'}a''_\beta-a'_\beta a''_{\alpha'}\}(\alpha'\beta x)\{a^{\mathrm{V}}_x b_\alpha-a^{\mathrm{V}}_\alpha b_x\}a_x(aa^{\mathrm{V}}b)=a'_{\alpha'}a''_\beta a^{\mathrm{V}}_x b_\alpha(\alpha'\beta x)a_x(aa^{\mathrm{V}}b)\\
&\qquad -a'_{\alpha'}a''_\beta a^{\mathrm{V}}_\alpha b_x(\alpha'\beta x)a_x(aa^{\mathrm{V}}b)-a'_\beta a''_{\alpha'}a^{\mathrm{V}}_x b_\alpha(\alpha'\beta x)a_x(aa^{\mathrm{V}}b)\\
&\qquad +a'_\beta a''_{\alpha'}a^{\mathrm{V}}_\alpha b_x(\alpha'\beta x)a_x(aa^{\mathrm{V}}b)\\
&=0-(a'a'''a^{\mathrm{IV}})(\widehat{a'''a^{\mathrm{IV}}\beta x})a'_{(a''a^{\mathrm{V}})}a''_\beta b_x a_x(aa^{\mathrm{V}}b)-0\\
&\qquad +(a''a'''a^{\mathrm{IV}})(\widehat{a'''a^{\mathrm{IV}}\beta x})a''_{(a^{\mathrm{V}}a')}a'_\beta b_x a_x(aa^{\mathrm{V}}b)\\
&=-\tfrac{1}{3}D(\widehat{\beta x a''a^{\mathrm{V}}})a''_\beta b_x a_x(aa^{\mathrm{V}}b)+\tfrac{1}{3}D(\widehat{\beta x a^{\mathrm{V}}a'})a'_\beta b_x a_x(aa^{\mathrm{V}}b)\\
&=-\tfrac{1}{3}D(a''_\beta a^{\mathrm{V}}_x-a''_x a^{\mathrm{V}}_\beta)a''_\beta b_x a_x(aa^{\mathrm{V}}b)+\tfrac{1}{3}D(a^{\mathrm{V}}_\beta a'_x-a^{\mathrm{V}}_x a'_\beta)a'_\beta b_x a_x(aa^{\mathrm{V}}b)\\
&=\tfrac{1}{3}Da_x a''_x a''_\beta a^{\mathrm{V}}_\beta b_x(aa^{\mathrm{V}}b)+\tfrac{1}{3}Da^{\mathrm{V}}_\beta a'_x a'_\beta b_x a_x(aa^{\mathrm{V}}b)\\
&=\tfrac{1}{6}Da''_x a''_\beta b_x(aa^{\mathrm{V}}b)\{a_x a^{\mathrm{V}}_\beta-a_\beta a^{\mathrm{V}}_x\}+\tfrac{1}{6}Da'_x a'_\beta b_x(aa^{\mathrm{V}}b)\{a^{\mathrm{V}}_\beta a_x-a^{\mathrm{V}}_x a_\beta\}\\
&=\tfrac{1}{6}Da''_x a''_\beta b_x b_\alpha(\alpha x\beta)-\tfrac{1}{6}Da'_x a'_\beta b_x b_\alpha(\alpha\beta x)=-\tfrac{1}{6}DG-\tfrac{1}{6}DG=-\tfrac{1}{3}DG.
\end{aligned}$$

6. $(bab''')(\alpha\alpha'\beta)(\widehat{\alpha ab'''x})b_x(\alpha'\beta x)$

$$\begin{aligned}
&=(a'_{\alpha'}a''_\beta-a'_\beta a''_{\alpha'})(\alpha'\beta x)(a_x b'''_\alpha-a_\alpha b'''_x)b_x(bab''')\\
&=a'_{\alpha'}a''_\beta a_x b'''_\alpha(\alpha'\beta x)b_x(bab''')-0-a'_\beta a''_{\alpha'}a_x b'''_\alpha(\alpha'\beta x)b_x(bab''')+0\\
&=2a'_{\alpha'}a''_\beta a_x b'''_\alpha(\alpha'\beta x)b_x(bab''')=2(a'a'''a^{\mathrm{IV}})(\widehat{a'''a^{\mathrm{IV}}\beta x})a'_{(a''b''')}a''_\beta a_x b_x(bab''')\\
&=\tfrac{2}{3}D(\widehat{\beta x a''b'''})a''_\beta a_x b_x(bab''')=\tfrac{2}{3}D\{a''_\beta b'''_x-a''_x b'''_\beta\}a''_\beta a_x b_x(bab''')\\
&=-\tfrac{2}{3}Da''_x b'''_\beta a''_\beta a_x b_x(bab''')=-\tfrac{2}{3}D(b'''b'b'')(b'b''a'')b'''_{(ba)}a''_x a_x b_x=0.
\end{aligned}$$

7. $a_{\beta'}(\widehat{a'''b\alpha\beta})(\widehat{a'''b\beta'x})a_x(\alpha\beta x)=a_{\beta'}a_x(a'''_\alpha b_\beta-a'''_\beta b_\alpha)(a'''_{\beta'}b_x-a'''_x b_{\beta'})(\alpha\beta x)$

$$\begin{aligned}
&=a_{\beta'}a_x(\alpha\beta x)[a'''_\alpha b_\beta a'''_{\beta'}b_x-a'''_\alpha b_\beta a'''_x b_{\beta'}-a'''_\beta b_\alpha a'''_{\beta'}b_x+a'''_\beta b_\alpha a'''_x b_{\beta'}]\\
&=(a'''a'a'')(\widehat{a'a''\beta x})a'''_{\beta'}a'_\beta a_x b_\beta b_x-(a'''a'a'')(\widehat{a'a''\beta x})a'''_x b_\beta b_{\beta'}a_{\beta'}a_x\\
&\qquad -a'''_\beta b_\alpha a'''_{\beta'}b_x a_{\beta'}a_x(\alpha\beta x)+(bb'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}a)b_\alpha a'''_x a'''_\beta a_x(\alpha\beta x)\\
&=\tfrac{1}{3}D(\beta x\beta')a_{\beta'}a_x b_\beta b_x-0-A_{122}G+\tfrac{1}{3}D(aa'a'')(\widehat{a'a''\beta x})a_x a'''_x a'''_\beta=-A_{122}G.
\end{aligned}$$

8. $b_{\beta'}(\widehat{ab'''\alpha\beta})(\widehat{ab'''\beta'x})b_x(\alpha\beta x)=(bb'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\overline{xab'''})b_x$, etc. $=0$.

9. $a_{\alpha'}(\widehat{a'''b\alpha\beta})(\widehat{a'''b\alpha'x})a_x(\alpha\beta x)=(aa'''a^{\mathrm{IV}})(\widehat{a'''a^{\mathrm{IV}}x\beta})a_x(\beta\alpha\beta')(\alpha\beta'x)$

$=\tfrac{1}{3}D(x\beta x)$, etc. $=0$.

10. $b_{\alpha'}(\widehat{ab'''\alpha\beta})(\widehat{ab'''\alpha'x})b_x(\alpha\beta x)=b_{\alpha'}b_x a_\beta b'''_\alpha a_x b'''_{\alpha'}(\alpha\beta x)$

But, $b_{\alpha'}b_x b'''_\alpha b'''_{\alpha'}a_\beta a_x(\alpha\beta x)-b'''^{2}_{\alpha'}b_x b_\alpha a_\beta a_x(\alpha\beta x)=b'''_{\alpha'}b_x a_\beta a_x(\alpha\beta x)[b_{\alpha'}b'''_\alpha-b_\alpha b'''_{\alpha'}]$

$$\begin{aligned}
&=b'''_{\alpha'}b_x a_\beta a_x(\alpha\beta x)(\widehat{bb'''\alpha'\alpha})=\tfrac{1}{2}a_x b_x a_\beta(\widehat{bb'''\alpha'\alpha})[(\alpha\beta x)\alpha'_{b'''}-(\alpha'\beta x)\alpha_{b'''}]\\
&=\tfrac{1}{2}a_x b_x a_\beta(\widehat{bb'''\alpha'\alpha})[(\alpha\beta\alpha')b'''_x-(\alpha x\alpha')b'''_\beta]=0.
\end{aligned}$$

$\therefore b_{\alpha'}(\widehat{ab'''\alpha\beta})(\widehat{ab'''\alpha'x})b_x(\alpha\beta x)=A_{112}G.$

11. $(aa^{\mathrm{IV}}b''')(\widehat{a'''b}\alpha\beta)(\widehat{a'''b}\,\widehat{a^{\mathrm{IV}}b'''}x)a_x(\alpha\beta x)=-\tfrac{1}{2}A_{112}G.$

For, $(aa^{\mathrm{IV}}b''')\{a_\alpha'''b_\beta-a_\beta'''b_\alpha\}(\alpha\beta x)a_x\{(a'''a^{\mathrm{IV}}b''')b_x-(ba^{\mathrm{IV}}b''')a_x'''\}$

$=(aa^{\mathrm{IV}}b''')\{a_\alpha'''b_\beta a_x(\alpha\beta x)-a_\beta'''b_\alpha a_x(\alpha\beta x)\}\{(a'''a^{\mathrm{IV}}b''')b_x-(ba^{\mathrm{IV}}b''')a_x'''\}.$

$(aa^{\mathrm{IV}}b''')a_\alpha'''b_\beta a_x(\alpha\beta x)(a'''a^{\mathrm{IV}}b''')b_x=0$

and $(aa^{\mathrm{IV}}b''')a_\alpha'''b_\beta a_x(\alpha\beta x)(ba^{\mathrm{IV}}b''')a_x'''=0.$

But, $-(aa^{\mathrm{IV}}b''')a_\beta'''b_\alpha a_x(\alpha\beta x)(a'''a^{\mathrm{IV}}b''')b_x$

$=-\tfrac{1}{2}b_\alpha a_x(\alpha\beta x)b_x(a'''a^{\mathrm{IV}}b''')[(aa^{\mathrm{IV}}b''')a_\beta'''-(aa'''b''')a_\beta^{\mathrm{IV}}]$

$=-\tfrac{1}{2}b_\alpha a_x(\alpha\beta x)b_x(a'''a^{\mathrm{IV}}b''')[(aa^{\mathrm{IV}}a''')b_\beta'''+(a^{\mathrm{IV}}b'''a''')a_\beta]$

$=\tfrac{1}{2}(aa'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}b''')a_xb_\alpha(\alpha\beta x)b_xb_\beta'''-\tfrac{1}{2}(a'''a^{\mathrm{IV}}ab''')^2b_\alpha a_xa_\beta b_x(\alpha\beta x)$

$=-\tfrac{1}{2}A_{112}G.$ Also $(aa^{\mathrm{IV}}b''')a_\beta'''b_\alpha a_x(\alpha\beta x)(ba^{\mathrm{IV}}b''')a_x'''$

$=\tfrac{1}{2}a_\beta'''a_xa_x'''(\alpha\beta x)(ba^{\mathrm{IV}}b''')\{(aa^{\mathrm{IV}}b''')b_\alpha-(aa^{\mathrm{IV}}b)b_\alpha'''\}$

$=\tfrac{1}{2}a_\beta'''a_xa_x'''(\alpha\beta x)(ba^{\mathrm{IV}}b''')\{(abb''')a_\alpha^{\mathrm{IV}}-(a^{\mathrm{IV}}b'''b)a_\alpha\}$

$=\tfrac{1}{2}a_\beta'''a_xa_x'''a_\alpha^{\mathrm{IV}}(\alpha\beta x)(ba^{\mathrm{IV}}b''')(abb''')+\tfrac{1}{2}(ba^{\mathrm{IV}}b''')^2a_\alpha a_\beta'''a_xa_x'''(\alpha\beta x)$

$=\tfrac{1}{2}(a^{\mathrm{V}}a'a'')(a'a''\widehat{\beta x})a^{\mathrm{IV}}_{(b'''b)}a_\beta'''a_xa_x'''(abb''')=\tfrac{1}{6}D(\widehat{\beta x}b'''b)a_\beta'''a_xa_x'''(abb''')$

$=\tfrac{1}{6}D\{b_\beta'''b_x-b_x'''b_\beta\}a_\beta'''a_xa_x'''(abb''')$

$=\tfrac{1}{6}D(b'''b'b'')(b'b''a''')b'''_{(ab)}b_xa_xa_x'''-\tfrac{1}{6}D(bb'b'')(b'b''a''')b_{(b'''a)}b_x''a_xa_x'''$

$=\tfrac{1}{18}DD'(a'''ab)a_xa_x'''b_x-\tfrac{1}{18}DD'(a'''b'''a)a_x'''a_xb_x'''=0.$

12. $(ba'''b^{\mathrm{IV}})(\widehat{ab'''}\alpha\beta)(\widehat{ab'''}\,\widehat{a'''b^{\mathrm{IV}}}x)b_x(\alpha\beta x)$

$=(ba'''b^{\mathrm{IV}})\{a_\alpha b_\beta'''-a_\beta b_\alpha'''\}\beta_{(x\alpha)}\{(aa'''b^{\mathrm{IV}})b_x'''-(b'''a'''b^{\mathrm{IV}})a_x\}b_x$

$=0-0-(ba'''b^{\mathrm{IV}})a_\beta b_\alpha'''(\alpha\beta x)(aa'''b^{\mathrm{IV}})b_x'''b_x$

$+(ba'''b^{\mathrm{IV}})a_\beta b_\alpha'''(\alpha\beta x)(b'''a'''b^{\mathrm{IV}})a_xb_x$

$=-\tfrac{1}{2}b_xb_x'''b_\alpha'''(\alpha\beta x)(aa'''b^{\mathrm{IV}})\{(ba'''b^{\mathrm{IV}})a_\beta-(bab^{\mathrm{IV}})a_\beta'''\}$

$+\tfrac{1}{2}a_\beta(\alpha\beta x)(b'''a'''b^{\mathrm{IV}})a_xb_x\{(ba'''b^{\mathrm{IV}})b_\alpha'''-(ba'''b''')b_\alpha^{\mathrm{IV}}\}$

$=-\tfrac{1}{2}b_xb_x'''b_\alpha'''(\alpha\beta x)(aa'''b^{\mathrm{IV}})\{(ba'''a)b_\beta^{\mathrm{IV}}+(a'''b^{\mathrm{IV}}a)b_\beta\}$

$+\tfrac{1}{2}a_\beta(\alpha\beta x)(b'''a'''b^{\mathrm{IV}})a_xb_x\{(bb'''b^{\mathrm{IV}})a_\alpha'''-(a'''b'''b^{\mathrm{IV}})b_\alpha\}$

$=0+0+0+\tfrac{1}{2}(a'''b'''b^{\mathrm{IV}})^2a_\beta a_xb_xb_\alpha(\alpha\beta x)=\tfrac{1}{2}A_{122}G.$

13. $a_{\beta''}(\beta\alpha\beta')(\beta\beta''x)a_x(\alpha\beta'x)=a_{\beta''}\{b_\alpha'b_{\beta'}''-b_{\beta'}'b_\alpha''\}\{b_{\beta''}'b_x''-b_x'b_{\beta''}''\}(\beta'x\alpha)a_x$

$=0-a_{\beta''}a_xb_\alpha'b_x'(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b_{\beta''}''$

$-a_{\beta''}a_x(b'b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b_{\beta''}'b_\alpha''b_x''+0$

$=-\tfrac{1}{3}a_{\beta''}a_xb_\alpha'b_x'D'(x\alpha\beta'')-\tfrac{1}{3}a_{\beta''}a_xb_\alpha''b_x''D'(x\alpha\beta'')=-\tfrac{2}{3}D'G.$

14. $b_{\beta''}(\beta\alpha\beta')(\beta\beta''x)b_x(\alpha\beta'x)=(bb^{\mathrm{V}}b^{\mathrm{VI}})(b^{\mathrm{V}}b^{\mathrm{VI}}\widehat{x\beta})b_x(\beta\alpha\beta')(\alpha\beta'x)=0.$

15. $a_{\alpha'}(\beta\alpha\beta')(\beta\alpha'x)a_x(\alpha\beta'x)=(aa'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}\widehat{x\beta})a_x(\beta\alpha\beta')(\alpha\beta'x)=0.$

16. $$\begin{aligned}
&b_{\alpha'}(\beta\alpha\beta')(\beta\alpha'x)b_x(\alpha\beta'x)=b_{\alpha'}\{b'_\alpha b''_{\beta'}-b'_{\beta'}b''_\alpha\}\{b'_{\alpha'}b''_x-b'_xb''_{\alpha'}\}(\beta'x\alpha)b_x\\
&=0-(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b''_{\alpha'}b'_\alpha b'_x b_x b_{\alpha'}\\
&\qquad-(b'b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b'_{\alpha'}b''_\alpha b''_x b_x b_{\alpha'}+0=-\tfrac{1}{3}D'(x\alpha\alpha')b'_\alpha b'_x b_x b_{\alpha'}\\
&\qquad-\tfrac{1}{3}D'(x\alpha\alpha')b''_\alpha b''_x b_x b_{\alpha'}=-\tfrac{2}{3}D'(\alpha\alpha'x)b'_\alpha b_{\alpha'}b'_x b_x\\
&=-\tfrac{2}{3}D'(a'_{\alpha'}a''_x-a'_xa''_{\alpha'})b'_\alpha b_{\alpha'}b_x b'_x=-\tfrac{2}{3}D'(a'a'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}b)a'_{(a''b')}a''_x b_x b'_x\\
&\qquad+\tfrac{2}{3}D'(a''a'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}b)a''_{(b'a')}b_x a'_x b'_x=-\tfrac{2}{9}DD'(ba''b')b_x b'_x a''_x\\
&\qquad+\tfrac{2}{9}DD'(bb'a')b_x b'_x a'_x=0.
\end{aligned}$$

17. $$\begin{aligned}
&(aa'''b)(\beta\alpha\beta')(\beta\widehat{a'''b}x)a_x(\alpha\beta'x)\\
&=(aa'''b)(\alpha\beta'x)a_x\{b'_\alpha b''_{\beta'}-b'_{\beta'}b''_\alpha\}\{a'''_x b_\beta-a'''_\beta b_x\}\\
&=(aa'''b)(\alpha\beta'x)a_x\{b'_\alpha b''_{\beta'}a'''_x b_\beta-b'_\alpha b''_{\beta'}a'''_\beta b_x-b'_{\beta'}b''_\alpha a'''_x b_\beta+b'_{\beta'}b''_\alpha a'''_\beta b_x\}\\
&=(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b''_{(bb')}a'''_x b'_\alpha a_x(aa'''b)\\
&\qquad-(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b''_{(a'''b')}b_x b'_\alpha a_x(aa'''b)\\
&\qquad-(b'b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b'_{(b''b)}a'''_x b''_\alpha a_x(aa'''b)\\
&\qquad+(b'b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b'_{(b''a''')}b_x b''_\alpha a_x(aa'''b)\\
&=\tfrac{1}{3}D'(bb'\widehat{x\alpha})b'_\alpha a_x a'''_x(aa'''b)-\tfrac{1}{3}D'(a'''b'\widehat{x\alpha})b_x b'_\alpha a_x(aa'''b)\\
&\qquad-\tfrac{1}{3}D'(b''b\widehat{x\alpha})b''_\alpha a_x a'''_x(aa'''b)+\tfrac{1}{3}D'(b''a'''\widehat{x\alpha})b_x b''_\alpha a_x(aa'''b)\\
&=-\tfrac{2}{3}D'\{a'''_x b'_\alpha-a'''_\alpha b'_x\}b_x b'_\alpha a_x(aa'''b)=-\tfrac{2}{3}D'b'^2_\alpha a'''_x b_x a_x(aa'''b)\\
&\qquad+\tfrac{2}{3}D'a'''_\alpha b'_x b_x b'_\alpha a_x(aa'''b)=\tfrac{2}{3}D'(a'''a'a'')(a'a''b')a'''_{(ba)}b'_x b_x a_x\\
&=\tfrac{2}{9}DD'(b'ba)b'_x b_x a_x=0.
\end{aligned}$$

18. $$\begin{aligned}
&(bab^{\mathrm{V}})(\beta\alpha\beta')(\beta\widehat{ab^{\mathrm{V}}}x)b_x(\alpha\beta'x)\\
&=(bab^{\mathrm{V}})(\beta\widehat{ab^{\mathrm{V}}}x)b_x\{b'_\alpha b''_{\beta'}-b'_{\beta'}b''_\alpha\}\{a'_{\beta'}a''_x-a'_xa''_{\beta'}\}\\
&=(bab^{\mathrm{V}})b_x[b'_\alpha b''_{\beta'}a'_{\beta'}a''_x-b'_\alpha b''_{\beta'}a'_xa''_{\beta'}-b'_{\beta'}b''_\alpha a'_{\beta'}a''_x+b'_{\beta'}b''_\alpha a'_xa''_{\beta'}](a_xb^{\mathrm{V}}_\beta-a_\beta b^{\mathrm{V}}_x)\\
&=(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}a')b''_{(b^{\mathrm{V}}b')}b'_\alpha a''_x a_x(bab^{\mathrm{V}})b_x\\
&\qquad-(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}a')b''_{(ab')}b'_\alpha a''_x b^{\mathrm{V}}_x(bab^{\mathrm{V}})b_x,\ \text{etc.}\\
&=\tfrac{1}{3}D'(a'b^{\mathrm{V}}b')b'_\alpha a''_x a_x b_x(bab^{\mathrm{V}})-\tfrac{1}{3}D'(a'ab')b'_\alpha a''_x b^{\mathrm{V}}_x b_x(bab^{\mathrm{V}}),\ \text{etc.}\\
&=\tfrac{1}{6}D'a''_x a_x b_x(a'b^{\mathrm{V}}b')[(bab^{\mathrm{V}})b'_\alpha-(bab')b^{\mathrm{V}}_\alpha]-0,\ \text{etc.}\\
&=\tfrac{1}{6}D'a''_x a_x b_x(a'b^{\mathrm{V}}b')[(ab^{\mathrm{V}}b')b_\alpha-(bb^{\mathrm{V}}b')a_\alpha],\ \text{etc.}\\
&=\tfrac{1}{6}D'a''_x a_x b_x(a'b^{\mathrm{V}}b')(ab^{\mathrm{V}}b')b_\alpha,\ \text{etc.}=\tfrac{1}{6}D'a''_x a_x b_x a'_\beta a_\beta b_\alpha,\ \text{etc.}\\
&=\tfrac{1}{12}D'a_x b_x a_\beta b_\alpha[a''_x a'_\beta-a'_xa''_\beta],\ \text{etc.}=\tfrac{1}{12}D'a_x b_x a_\beta b_\alpha(\alpha\beta x),\ \text{etc.}\\
&=\tfrac{1}{12}D'G+\tfrac{1}{12}D'G+\tfrac{1}{12}D'G+\tfrac{1}{12}D'G=\tfrac{1}{3}D'G.
\end{aligned}$$

Combining these results as in (3) we obtain G'.

TABLE II.

System of $a_x^2+kb_x^2$, $(\alpha\beta x)^2$.

1. $K \;=a_x^2+kb_x^2.$

2. $F \;=(\alpha\beta x)^2.$

3. $(L)'=\alpha_u^2+2k(abu)^2+k^2\beta_u^2.$

4. $(L')'=\frac{4}{9}\{3DA_{122}u_\beta^2+3D'A_{112}u_\alpha^2-2DD'(abu)^2\}.$

5. $(D)'=D+3kA_{112}+3k^2A_{122}+k^3D'.$

6. $(D')'=\frac{8}{27}\{9A_{112}A_{122}-DD'\}DD'.$

7. $\Phi' =\frac{1}{3}DL'+A_{122}L+k\{\frac{1}{3}D'L+A_{112}L'\}.$

8. $C_1' =C_8-kC_5.$

9. $A_{122}' =\frac{4}{9}\{3DA_{122}^2+DD'A_{112}+k(3D'A_{112}^2+DD'A_{122})\}.$

10. $C_2' ={}_4\{3DA_{122}C_2+DD'C_3+k(3D'A_{112}C_3+DD'C_2)\}.$

11. $C_3' =(\frac{2}{3}DA_{122}u_x-\frac{2}{3}DC_2)+2k(\frac{1}{9}DD'u_x-A_{112}C_2+A_{112}A_{122}u_x-A_{122}C_3)$
$$+k^2(\tfrac{2}{3}D'A_{112}u_x-\tfrac{2}{3}D'C_3).$$

12. $A_{112}' =\frac{4}{3}DA_{122}+2k(\frac{1}{3}DD'+A_{112}A_{122})+k^2\cdot\frac{4}{3}D'A_{112}.$

13. $C_4' =\frac{4}{9}[3DA_{122}C_4-2DD'C_7+6k(DA_{122}C_6-D'A_{112}C_7)$
$$-k^2(3D'A_{112}C_4-2DD'C_6)].$$

14. $F' =\frac{4}{9}\{3DA_{122}F+2DD'A_{112}f-\frac{2}{3}D^2D'g$
$$+2k(3DA_{122}^2g-DD'A_{122}f-A_{112}DD'g+3A_{112}^2D'f+DD'F)$$
$$+k^2(3D'A_{112}F+2DD'A_{122}g-\tfrac{2}{3}DD'^2f)\}.$$

15. $C_5' =\frac{4}{9}\{2DD'A_{112}C_8+\frac{2}{3}D^2D'C_5$
$$+2k(3D'A_{112}^2C_8+DD'A_{112}C_5-3DA_{122}^2C_5-DD'A_{122}C_8)$$
$$+k^2(-2DD'A_{122}C_5-\tfrac{2}{3}DD'^2C_8)\}.$$

16. $C_6' =\frac{4}{9}\{3DA_{122}^2C_4-2DD'A_{122}C_7-DD'A_{112}C_4+\frac{2}{3}D^2D'C_6$
$$+k(-3D'A_{112}^2C_4+2DD'A_{112}C_6+DD'A_{122}C_4-\tfrac{2}{3}DD'^2C_7)\}.$$

17. $C_7' =\frac{1}{3}DC_4+k(A_{112}C_4-2A_{122}C_7+\frac{2}{3}DC_6)$
$$+k^2(-A_{122}C_4+A_{112}C_6-\tfrac{2}{3}D'C_7)+k^3(-\tfrac{1}{3}D'C_4).$$

18. $C_8' =\frac{4}{9}\{3DA_{122}C_8-\frac{2}{3}D^2D'C_1+k(6DA_{122}^2C_1-4DD'A_{112}C_1+2DD'C_8-3DA_{122}C_5)$
$$+k^2(-6D'A_{112}^2C_1+4DD'A_{122}C_1-2DD'C_5+3D'A_{112}C_8)$$
$$+k^3(\tfrac{2}{3}DD'^2C_1-3D'A_{112}C_5)\}.$$

19. $G' =\frac{4}{9}\{\frac{2}{3}D^2D'+k(4DD'A_{112}-6DA_{122}^2)$
$$+k^2(6D'A_{112}^2-4DD'A_{122})-\tfrac{2}{3}DD'^2\cdot k^3\}G.$$

20. $\Gamma' =\frac{4}{9}\{\frac{2}{3}D^2D'+k(4DD'A_{112}-6DA_{122}^2)$
$$+k^2(6D'A_{112}^2-4DD'A_{122})-\tfrac{2}{3}DD'^2\cdot k^3\}\Gamma.$$

§ 3. SYSTEM OF $a_x^2+kb_x^2$, c_x^2.

TABLE III.

$(a_x^2+kb_x^2, c_x^2)_{kl}^{ij}$ *in terms of the system of Baker.*

$$
\begin{aligned}
\Phi'' &= (a_x^2+kb_x^2, c_x^2)_{00}^{02}=\Phi_2+k\Phi_1.\\
C_1'' &= (a_x^2+kb_x^2, c_x^2)_{00}^{01}=C_{12}+kC_{11}.\\
A_{122}'' &= (a_x^2+kb_x^2, \gamma_u^2)_{00}^{20}=A_1+kA_2.\\
C_2'' &= (a_x^2+kb_x^2, \gamma_u^2)_{00}^{10}=C_{21}+kC_{22}.\\
C_3'' &= [\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2, c_x^2]_{10}^{00}=C_{23}+2kM+k^2C_{24}.\\
A_{112}'' &= [\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2, c_x^2]_{20}^{00}=A_3+2kN+k^2A_4.\\
C_4'' &= [\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2, \gamma_u^2]_{01}^{00}=C_{42}+2k(O_1-O_2)+k^2C_{41}.\\
F'' &= [\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2, \gamma_u^2]_{02}^{00}=F_2+2k(A_1g-2P+A_2f)+k^2F_1.\\
C_5'' &= [\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2, c_x^2\gamma_u^2]_{11}^{00}=C_{51}+2k(Q_1-Q_2)+k^2C_{52}.\\
C_6'' &= (a_x^2+kb_x^2, c_x^2\gamma_u^2)_{00}^{11}=C_{61}+kC_{62}.\\
C_7'' &= [(a_x^2+kb_x^2)(\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2), c_x^2]_{10}^{01}\\
&=C_{63}+k(2R_1-S_1J+2O_3)+k^2(2R_2-S_2J+2O_4)+k^3C_{64}.\\
C_8'' &= [(a_x^2+kb_x^2)(\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2), \gamma_u^2]_{01}^{10}\\
&=C_{53}+k(2A_1C_1-2fS_3+3T_1-T_2-JU_1)\\
&\qquad\qquad+k^2(-2A_2C_1+2gS_3+3T_3-T_4-JU_2)+k^3C_{54}.\\
G'' &= [(a_x^2+kb_x^2)(\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2), c_x^2\gamma_u^2]_{11}^{10}\\
&=G_1+k[2A_1V-2fW+X_1]+k^2[-2A_2V+2gW+X_2]+k^3G_2.\\
\Gamma'' &= [(a_x^2+kb_x^2)(\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2), c_x^2\gamma_u^2]_{10}^{11}\\
&=\Gamma_2+k[2A_1Y+S_1L''-2S_3\Phi_2+Z_1]+k^2[-2A_2Y-S_2L''+2S_3\Phi_1+Z_2]+k^3\Gamma_1.
\end{aligned}
$$

Note on Seminvariants of Systems of Partial Differential Equations.

By A. L. Nelson.

1. *Introduction.*

In the discussion of the projective differential geometry of a geometric configuration by means of Wilczynski's method, one of the necessary steps is the construction of a fundamental set of seminvariants, that is, such a set that any seminvariant whatever may be expressed in terms of them and their derivatives. In the cases where the completely integrable system of partial differential equations employed has one dependent variable, this construction has been accomplished by the reduction of the system of equations to its canonical form, the independent coefficients of which form are the fundamental set required. Let us illustrate the process by the case of plane nets.* The completely integrable system of equations for this case is the following:

$$y_{uu}=ay_u+by_v+cy, \quad y_{uv}=a'y_u+b'y_v+c'y, \quad y_{vv}=a''y_u+b''y_v+c''y. \tag{1}$$

The transformation

$$y=\lambda\bar{y} \tag{2}$$

yields a new system of equations of the same form as (1), with the coefficients

$$\left.\begin{aligned}
&\bar{a}=a-2\frac{\lambda_u}{\lambda}, && \bar{b}=b, && \bar{c}=c+a\frac{\lambda_u}{\lambda}+b\frac{\lambda_v}{\lambda}-\frac{\lambda_{uu}}{\lambda},\\
&\bar{a}'=a'-\frac{\lambda_v}{\lambda}, && \bar{b}'=b'-\frac{\lambda_u}{\lambda}, && \bar{c}'=c'+a'\frac{\lambda_u}{\lambda}+b'\frac{\lambda_v}{\lambda}-\frac{\lambda_{uv}}{\lambda},\\
&\bar{a}''=a'', && \bar{b}''=b''-2\frac{\lambda_v}{\lambda}, && \bar{c}''=c''+a''\frac{\lambda_u}{\lambda}+b''\frac{\lambda_v}{\lambda}-\frac{\lambda_{vv}}{\lambda}.
\end{aligned}\right\} \tag{3}$$

The integrability conditions enable us to find a function $p(u, v)$, such that $p_u=a+b'$, $p_v=a'+b''$. We find also that under the transformation (2) these combinations become

$$\overline{p_u}=p_u-3\frac{\lambda_u}{\lambda}, \quad \overline{p}=p_v-3\frac{\lambda_v}{\lambda}.$$

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* Cf. E. J. Wilczynski, "One-Parameter Families and Nets of Plane Curves," *Trans. Amer. Math. Soc.*, Vol. XII (1911), pp. 473–510.

If, therefore, λ be so chosen that

$$\frac{\lambda_u}{\lambda}=\tfrac{1}{3}p_u,\quad \frac{\lambda_v}{\lambda}=\tfrac{1}{3}p_v,$$

we get the special unique form of (1), which shall be indicated by capital letter coefficients, characterized by the relations $A+B'=0$, $A'+B''=0$. This is the canonical form of (1). We list its coefficients for later comparison.

$$\left.\begin{array}{lll}
A=\tfrac{1}{3}(a-2b'), & B=b, & C=c+\tfrac{2}{9}a^2+\tfrac{1}{9}ab'+\tfrac{1}{3}a'b+\tfrac{1}{3}bb'' \\
 & & \qquad -\tfrac{1}{3}a_u-\tfrac{1}{3}b'_u-\tfrac{1}{9}b'^2, \\
A'=\tfrac{1}{3}(2a'-b''), & B'=\tfrac{1}{3}(2b'-a), & C'=c'+\tfrac{2}{9}aa'+\tfrac{5}{9}a'b'+\tfrac{2}{9}b'b'' \\
 & & \qquad -\tfrac{1}{9}ab''-\tfrac{1}{3}a_v-\tfrac{1}{3}b'_v, \\
A''=a'', & B''=\tfrac{1}{3}(b''-2a'), & C''=c''+\tfrac{2}{9}b''^2+\tfrac{1}{9}a'b''+\tfrac{1}{3}a''b'+\tfrac{1}{3}aa'' \\
 & & \qquad -\tfrac{1}{3}b''_v-\tfrac{1}{3}a'_v-\tfrac{1}{9}a'^2.
\end{array}\right\}\quad (4)$$

Only seven of these coefficients are independent. They are the fundamental seminvariants. For, consider any system of equations of the form (1), to which we shall refer as (s), and any other system (t), obtained from it by any transformation of the type (2). Let (s) be reduced to its canonical form (S), and (t) to its canonical form (T). The coefficients of (S) are the same combinations of the coefficients of (s) as the corresponding coefficients of (T) are of those of (t). But, since the canonical form is unique, the corresponding coefficients of (S) and (T) are identical. Hence they are seminvariants. Moreover, any seminvariant whatever, I, is a function of the original coefficients of (s), *i. e.*, $I=I(a, b, \ldots, c'', a_u, a_v, \ldots)$. Since it is a seminvariant, it must be identical with the same function of the corresponding coefficients of any system of equations obtained from (s) by a transformation (2). In particular, $I=I(A, B, \ldots, C'', A_u, A_v, \ldots)$. Therefore I is a function of the seven independent coefficients of the canonical form and of their derivatives. Hence these coefficients are a fundamental set of seminvariants.

The determination of this canonical form depends upon the ability to find such a function p. Green* has shown that this can be done for completely integrable systems with one dependent variable and n independent variables, provided certain conditions are fulfilled. However, the expressions for p_u and p_v, which, with their derivatives, must occur in the coefficients of the canonical

*G. M. Green, "The Linear Dependence of Functions of Several Variables, and Completely Integrable Systems of Homogeneous Linear Partial Differential Equations," *Trans. Amer. Math. Soc.*, Vol. XVII (1916), pp. 483–516.

form, are not always simple, and as a result the seminvariants which come from this regulation method are sometimes rather complicated when expressed in terms of the original coefficients. We proceed to indicate how (in most cases, at least) an alternative set of simpler seminvariants may be computed.

2. *The Effect of Transformation* (2) *in the General Case.*

We assume that the completely integrable system of equations (corresponding to (1)), to which we shall refer as (1′), has one dependent variable, y,* and n independent variables, $u_1, \ldots, u_n$, $(n>1)$. It will then consist of a certain number, p, of equations which express certain p derivatives of y in terms of q chosen derivatives (which Green has called the primary derivatives). Let us also assume that among the q primary derivatives are none of order higher than the order of the lowest of the p derivatives in the left members of the p equations.

The transformation (2) gives rise to the following expressions of the various derivatives of y in terms of the derivatives of $\bar{y}$.

$$\left.\begin{array}{l} y=\lambda\bar{y}, \\ y_{u_i}=\lambda\bar{y}_{u_i}+\lambda_{u_i}\bar{y}, \quad (i=1,2,\ldots,n), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ y_{u_1\ldots u_n}^{l_1\ldots l_n}=\displaystyle\sum_{\substack{p_1=0,\ldots,l_1\\ \cdots\cdots\\ p_n=0,\ldots,l_n}}\binom{l_1}{p_1}\cdots\binom{l_n}{p_n}\lambda_{u_1\ldots u_n}^{p_1\ldots p_n}\bar{y}_{u_1\ldots u_n}^{l_1-p_1\ldots l_n-p_n}, \end{array}\right\} \qquad (2')$$

where

$$\binom{l_j}{p_j}=\frac{l_j!}{p_j!\,(l_j-p_j)!} \quad\text{and}\quad y_{u_1\ldots u_n}^{l_1\ldots l_n}=\frac{\partial^{l_1+\cdots+l_n}y}{\partial u_1^{l_1}\ldots\partial u_n^{l_n}}.$$

If we substitute (2′) in the system (1′), and arrange the resulting equations properly, we get a new system of the same form as (1′), in which the dependent variable is $\bar{y}$. The coefficients of this new system, which, in collected form, we shall refer to as (3′), are equal to the corresponding coefficients of (1′) plus linear combinations of fractions whose numerators are derivatives of λ, and whose common denominator is λ. Each coefficient (3′) receives additional terms from those y-derivatives in its equation which come by differentiation from the y-derivative belonging to this coefficient. (Thus, for example, in (3), $\bar{b}'$, the coefficient of $\bar{y}_v$, can receive only one additional term, $-\frac{\lambda_u}{\lambda}$, which comes from y_{uv}; $\bar{c}'$ receives one λ-fraction from each of the derivatives y_u, y_v, y_{uv}.)

* This assumption is merely for the sake of simplifying the discussion. The process may be readily extended to cover the cases where the number of dependent variables is greater than one.

We arrange these coefficients (3′) in classes as follows: In the zero-th class shall be placed those coefficients in each row which are coefficients of those primary derivatives having the same order as the derivative in the left member of the equation belonging to that row. The i-th class shall consist of the coefficients of those primary derivatives in each row of order i less than the derivative in the left member of the equation belonging to that row. (Thus, in table (3), the coefficients in the first and second columns are of the first class; those in the third column are of the second class.) The zero-th class coefficients, if present, must be seminvariants, since their y-derivatives can not yield by differentiation any other y-derivative in their equations. The first class coefficients will be altered, if at all, by the addition of multiples of λ-fractions of the first order, the coefficients of which λ-fractions are integers or integral multiples of zero-th class coefficients (3′). In general, the i-th class coefficients will be altered by the addition of linear combinations of λ-fractions of order not higher than i. Each one of the coefficients of these λ-fractions will be an integer or an integral multiple of some one coefficient (3′) of classes 0 to $i-1$.

3. *Pseudo-Canonical Forms of* (1′).

Now it very frequently happens that *under suitable restrictions* we may make n first class coefficients (3′) vanish by a proper choice of λ. For example, in (3), under the assumption $a'_u = b'_v$, we might have made $\bar{a}' = \bar{b}' = 0$. Or we might have made $\bar{a} = \bar{b}'' = 0$, provided $a_v = b''_u$. If we can do this, under the necessary restriction the new form of the completely integrable system determined by this choice of λ is unique, because of the nature of the equations in λ. We shall call this unique form a *pseudo-canonical form of* (1′). Moreover, the necessary restrictions are always seminvariantive ones. For, consider certain n such first class coefficients,

$$\bar{\alpha}_i = \alpha_i - k_i \frac{\partial}{\partial u_i} \log \lambda, \qquad (i = 1, 2, \ldots, n), \tag{5}$$

where the k_i are at worst integral multiples of seminvariant coefficients of (1′). In order to choose λ to satisfy the equations

$$\alpha_i - k_i \frac{\partial}{\partial u_i} \log \lambda = 0, \qquad (i = 1, 2, \ldots, n), \tag{6}$$

the assumptions necessary are

$$\frac{\partial}{\partial u_i}\left(\frac{\alpha_j}{k_j}\right) = \frac{\partial}{\partial u_j}\left(\frac{\alpha_i}{k_i}\right), \qquad (i, j = 1, 2, \ldots, n), \tag{7}$$

which are easily seen to be seminvariantive from (5).

Let us assume the necessary restrictions (7) and choose λ in accordance with (6). However, there is a precaution to be observed in substituting for λ. Whenever a cross-derivative,

$$\frac{1}{\lambda}\cdot\frac{\partial^{p_1+\cdots\cdots+p_n}\lambda}{\partial u_1^{p_1}\ldots\partial u_n^{p_n}}\quad \text{(at least two } p\text{'s} \neq 0\text{)},$$

occurs, equations (7) afford us a choice in the substitution. We must, throughout a coefficient (3′), replace such cross-derivatives by the proper derivatives of the *same* members of equations (7). The reason for this precaution will become apparent in the next section. Then, since the restrictions are seminvariantive, and since the pseudo-canonical form is unique, the coefficients of this form are seminvariants, by precisely the same argument as was made in Section 1 for the coefficients of the canonical form in the case of plane nets.

4. *The Coefficients of a Pseudo-Canonical Form are a Fundamental Set of Seminvariants.*

Now, if the restrictions (7) be removed, the pseudo-canonical form will no longer exist, *but the combinations we have obtained are still seminvariants.* For, let us form the same combinations of the coefficients (3′). If we do this under the assumption (7), of course all traces of λ will disappear. That this must also be true without these assumptions is evident from the following considerations:

Let any one of these combinations be denoted by $f(a, b, \ldots.)$. It is composed of coefficients of (1′) of various classes. However, from the nature of the choice of λ, the only derivatives which appear in $f(a, b, \ldots.)$ are derivatives of coefficients of the first class. Now, let us form the same combination of the coefficients (3′), $f(\bar{a}, \bar{b}, \ldots.)$, and examine the resulting expression. It may acquire derivatives of coefficients of (1′) only from the following sources: 1. Those derivatives which appeared in $f(a, b, \ldots.)$ will remain. These, as we have noted, are derivatives of coefficients of (1′) of the first class only. 2. Those arguments, $\bar{a}, \bar{b}, \ldots.$, which are of the first class (whose expressions are given by (3′)), may have as coefficients of λ-fractions zero-th class coefficients of (1′). Hence, when the expressions (3′) for these arguments are substituted in $f(\bar{a}, \bar{b}, \ldots.)$, the resulting expression may contain derivatives of these zero-th class coefficients. It thus appears that the only derivatives of coefficients of (1′) of any class except the zero-th which

may occur are those from the first source. Moreover, we have taken the precaution that among these derivatives in $f(a, b, \ldots .)$ shall appear only one member of any of equations (7) and its derivatives. Hence, if we collect the expression resulting from substitution of the values (3′) for the arguments in $f(\bar{a}, \bar{b}, \ldots .)$, the collection being with respect to the λ-fractions of various orders, no one of these fractions may have as coefficient an expression which vanishes by virtue of assumptions (7). Therefore, since in the resulting expression all terms in λ cancelled under assumptions (7), the same thing must happen without these assumptions.

It is to be observed that in a pseudo-canonical form, there are $pq-n$ non-vanishing coefficients, precisely the number of independent coefficients in Green's generalized canonical form. Moreover, these coefficients are independent, since no two arise from the same coefficient of the original system (1′). From this it may be argued that the seminvariant coefficients of a pseudo-canonical form are a fundamental set.

5.* *Seminvariants Arising from Cross-Derivatives of log λ.*

In certain cases there is a lack of definiteness in the formation of seminvariants by the process outlined above. As mentioned previously, whenever a λ-fraction involving a cross-derivative of λ occurs in a coefficient (3′), either member of the proper equation of (7) may be used in forming the seminvariant corresponding to this coefficient. Indeed, certain expressions of the type

$$\frac{m \dfrac{\partial}{\partial u_i}\left(\dfrac{\alpha_j}{k_j}\right) + n \dfrac{\partial}{\partial u_j}\left(\dfrac{\alpha_i}{k_i}\right)}{m+n}$$

may be used instead of $\dfrac{\partial}{\partial u_i}\left(\dfrac{\alpha_j}{k_j}\right)$ or $\dfrac{\partial}{\partial u_j}\left(\dfrac{\alpha_i}{k_i}\right)$.

A coefficient of the type in question may be put in the form

$$a_0 + \sum_{\substack{i,j=1,\ldots,n \\ i<j}} a^{(i,j)} D^{(i,j)} + \text{sum of terms of type } a D^{(i,j)}_{u_1^{l_1} \ldots u_n^{l_n}}, (l_1, \ldots, l_n = 0, 1, \ldots .),$$

where $D^{(i,j)} \equiv \dfrac{\partial^2}{\partial u_i \partial u_j} \log \lambda$. Let us denote one of the possible substitutions for $D^{(i,j)}$ by $[D^{(i,j)}]$, and the other by $\{D^{(i,j)}\}$. We wish to find what expressions of the form

$$F^{(i,j)} \equiv \frac{m^{(i,j)} [D^{(i,j)}] + n^{(i,j)} \{D^{(i,j)}\}}{m^{(i,j)} + n^{(i,j)}}, \qquad (m^{(i,j)} + n^{(i,j)} \neq 0), \tag{8}$$

---

* Section 5 was added at Professor Wilczynski's suggestion.

may be used to replace $D^{(i,j)}$. We shall find it convenient to use the notation

$$(M^{(i,j)}_{\substack{p_1 \ldots p_n \\ u_1 \ldots u_n}} D^{(i,j)}_{\substack{q_1 \ldots q_n \\ u_1 \ldots u_n}}) \equiv m^{(i,j)}_{\substack{p_1 \ldots p_n \\ u_1 \ldots u_n}} [D^{(i,j)}_{\substack{q_1 \ldots q_n \\ u_1 \ldots u_n}}] + n^{(i,j)}_{\substack{p_1 \ldots p_n \\ u_1 \ldots u_n}} \{D^{(i,j)}_{\substack{q_1 \ldots q_n \\ u_1 \ldots u_n}}\}, \tag{9}$$

and it is easily verified that

$$(M^{(i,j)} D^{(i,j)})_{u_k} = (M^{(i,j)} D^{(i,j)}_{u_k}) + (M^{(i,j)}_{u_k} D^{(i,j)}), \text{ etc.}, \qquad (i, j, k = 1, \ldots, n).$$

Now, if we denote by $\bar{\theta} = \theta + \delta\theta$ the same combination of the transformed coefficients (3′) as θ is of the original coefficients of (1′), we see from (5) that $\overline{[D^{(i,j)}]} = [D^{(i,j)}] - D^{(i,j)}$, $\overline{\{D^{(i,j)}\}} = \{D^{(i,j)}\} - D^{(i,j)}$. If, therefore, $F^{(i,j)}$ is to replace $D^{(i,j)}$, it must happen that

$$\delta F^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}} + D^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}} = 0, \qquad (l_1, \ldots, l_n = 0, 1, \ldots).$$

In order to compute $\delta F^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}}$, let us successively differentiate the equations

$$(m^{(i,j)} + n^{(i,j)}) F^{(i,j)} = (M^{(i,j)} D^{(i,j)}),$$

which result from (8) and (9). We obtain

$$\sum_{\substack{p_1 = 0, \ldots, l_1 \\ \cdots\cdots \\ p_n = 0, \ldots, l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} (m^{(i,j)} + n^{(i,j)})_{\substack{p_1 \ldots p_n \\ u_1 \ldots u_n}} F^{(i,j)}_{\substack{l_1 - p_1 \ldots l_n - p_n \\ u_1 \ldots u_n}}$$

$$= \sum_{\substack{p_1 = 0, \ldots, l_1 \\ \cdots\cdots \\ p_n = 0, \ldots, l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} (M^{(i,j)}_{\substack{p_1 \ldots p_n \\ u_1 \ldots u_n}} D^{(i,j)}_{\substack{l_1 - p_1 \ldots l_n - p_n \\ u_1 \ldots u_n}}).$$

From these equations

$$F^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}} = \frac{1}{m^{(i,j)} + n^{(i,j)}} \Bigg\{ \sum_{\substack{p_1 = 0, \ldots, l_1 \\ \cdots\cdots \\ p_n = 0, \ldots, l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} (M^{(i,j)}_{\substack{p_1 \ldots p_n \\ u_1 \ldots u_n}} D^{(i,j)}_{\substack{l_1 - p_1 \ldots l_n - p_n \\ u_1 \ldots u_n}})$$

$$- \sum_{s=1}^{n} \sum_{\substack{p_r = 1, \ldots, l_r \\ r \neq s}} \binom{l_1}{p_1} \cdots \binom{l_{s-1}}{p_{s-1}} \binom{l_{s+1}}{p_{s+1}} \cdots \binom{l_n}{p_n} (m^{(i,j)} + n^{(i,j)})_{\substack{p_1 \ldots p_{s-1} \; p_{s+1} \ldots p_n \\ u_1 \ldots u_{s-1} \; u_{s+1} \ldots u_n}} F^{(i,j)}_{\substack{l_1 - p_1 \ldots l_s \ldots l_n - p_n \\ u_1 \ldots u_s \ldots u_n}}$$

$$- \sum_{\substack{p_1 = 1, \ldots, l_1 \\ \cdots\cdots \\ p_n = 1, \ldots, l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} (m^{(i,j)} + n^{(i,j)})_{\substack{p_1 \ldots p_n \\ u_1 \ldots u_n}} F^{(i,j)}_{\substack{l_1 - p_1 \ldots l_n - p_n \\ u_1 \ldots u_n}} \Bigg\},$$

and $\overline{F^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}}}$ will have a similar expression with $\overline{m^{(i,j)}}$, $\overline{n^{(i,j)}}$, $\overline{M^{(i,j)}}$, $\overline{D^{(i,j)}}$ and $\overline{F^{(i,j)}}$, replacing $m^{(i,j)}$, $n^{(i,j)}$, $M^{(i,j)}$, $D^{(i,j)}$ and $F^{(i,j)}$, respectively. Use of the relation

$$\delta F^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}} = \overline{F^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}}} - F^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}}$$

gives us, after considerable labor,

$$
\left.
\begin{aligned}
&\delta F^{(i,j)}_{\substack{l_1 \dots l_n \\ u_1 \dots u_n}} + D^{(i,j)}_{\substack{l_1 \dots l_n \\ u_1 \dots u_n}} = \\
&\text{(a)} \quad \sum_{\substack{p_1=0,\,\dots,\,l_1 \\ \dots\dots\dots \\ p_n=0,\,\dots,\,l_n}} \binom{l_1}{p_1}\cdots\binom{l_n}{p_n} \Bigg(\delta \frac{m^{(i,j)}_{\substack{p_1 \dots p_n \\ u_1 \dots u_n}}}{m^{(i,j)}+n^{(i,j)}} \left[D^{(i,j)}_{\substack{l_1-p_1 \, l_n-p_n \\ u_1 \dots u_n}} \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \delta \frac{n^{(i,j)}_{\substack{p_1 \dots p_n \\ u_1 \dots u_n}}}{m^{(i,j)}+n^{(i,j)}} \left\{ D^{(i,j)}_{\substack{l_1-p_1 \, l_n-p_n \\ u_1 \dots u_n}} \right\} \Bigg) \\
&\text{(b)} \quad -\sum_{s=1}^{n} \sum_{\substack{p_r=1,\,\dots,\,l_r \\ r \neq s}} \binom{l_1}{p_1}\cdots\binom{l_{s-1}}{p_{s-1}}\binom{l_{s+1}}{p_{s+1}}\cdots\binom{l_n}{p_n} \delta \frac{(m^{(i,j)}+n^{(i,j)})_{\substack{p_1 \dots p_{s-1} \, p_{s+1} \dots p_n \\ u_1 \dots u_{s-1} \, u_{s+1} \dots u_n}}}{m^{(i,j)}+n^{(i,j)}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot F^{(i,j)}_{\substack{l_1-p_1 \, l_s \, l_n-p_n \\ u_1 \dots u_s \dots u_n}} \\
&\text{(c)} \quad -\sum_{\substack{p_1=1,\,\dots,\,l_1 \\ \dots\dots\dots \\ p_n=1,\,\dots,\,l_n}} \binom{l_1}{p_1}\cdots\binom{l_n}{p_n} \delta \frac{(m^{(i,j)}+n^{(i,j)})_{\substack{p_1 \dots p_n \\ u_1 \dots u_n}}}{m^{(i,j)}+n^{(i,j)}} \cdot F^{(i,j)}_{\substack{l_1-p_1 \, l_n-p_n \\ u_1 \dots u_n}} \\
&\text{(d)} \quad -\sum_{s=1}^{n} \sum_{\substack{p_r=1,\,\dots,\,l_r \\ r \neq s}} \binom{l_1}{p_1}\cdots\binom{l_{s-1}}{p_{s-1}}\binom{l_{s+1}}{p_{s+1}}\cdots\binom{l_n}{p_n} \frac{(\overline{m^{(i,j)}}+\overline{n^{(i,j)}})_{\substack{p_1 \dots p_{s-1} \, p_{s+1} \dots p_n \\ u_1 \dots u_{s-1} \, u_{s+1} \dots u_n}}}{\overline{m^{(i,j)}}+\overline{n^{(i,j)}}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left(\delta F^{(i,j)}_{\substack{l_1-p_1 \, l_s \, l_n-p_n \\ u_1 \dots u_s \dots u_n}} + D^{(i,j)}_{\substack{l_1-p_1 \, l_s \, l_n-p_n \\ u_1 \dots u_s \dots u_n}} \right) \\
&\text{(e)} \quad -\sum_{\substack{p_1=1,\,\dots,\,l_1 \\ \dots\dots\dots \\ p_n=1,\,\dots,\,l_n}} \binom{l_1}{p_1}\cdots\binom{l_n}{p_n} \frac{(\overline{m^{(i,j)}}+\overline{n^{(i,j)}})_{\substack{p_1 \dots p_n \\ u_1 \dots u_n}}}{\overline{m^{(i,j)}}+\overline{n^{(i,j)}}} \left(\delta F^{(i,j)}_{\substack{l_1-p_1 \, l_n-p_n \\ u_1 \dots u_n}} + D^{(i,j)}_{\substack{l_1-p_1 \, l_n-p_n \\ u_1 \dots u_n}} \right).
\end{aligned}
\right\} \quad (10)
$$

It is evident that (a) will vanish if

$$
\delta \frac{m^{(i,j)}_{\substack{p_1 \dots p_n \\ u_1 \dots u_n}}}{m^{(i,j)}+n^{(i,j)}} = \delta \frac{n^{(i,j)}_{\substack{p_1 \dots p_n \\ u_1 \dots u_n}}}{m^{(i,j)}+n^{(i,j)}} = 0 \quad (p_1=0, \dots, l_1; \dots; p_n=0, \dots, l_n). \quad (11)
$$

The conditions (11) imply

$$
\delta \frac{(m^{(i,j)}+n^{(i,j)})_{\substack{p_1 \dots p_n \\ u_1 \dots u_n}}}{m^{(i,j)}+n^{(i,j)}} = 0 \quad (p_1=0, \dots, l_1; \dots; p_n=0, \dots, l_n),
$$

so that (b) and (c) vanish as a result. Moreover, (11) also imply

$$
\delta F^{(i,j)}_{\substack{l_1-p_1 \, l_s \, l_n-p_n \\ u_1 \dots u_s \dots u_n}} + D^{(i,j)}_{\substack{l_1-p_1 \, l_s \, l_n-p_n \\ u_1 \dots u_s \dots u_n}} = 0,
$$

($p_1=0, \dots, l_1$; $\dots$; $p_n=0, \dots, l_n$; except that not all p's$=0$; $s=1, \dots, n$), by virtue of (10) for successive values of the l's. Hence (d) and (e) also vanish as a result of (11). Therefore, *a sufficient condition that* $F^{(i,j)}$ *may*

replace $[D^{(i,j)}]$ *or* $\{D^{(i,j)}\}$ *in any seminvariant without affecting its seminvariance, is that for each pair* $(i, j=1, \ldots, n;\ i<j)$,

$$\frac{\overset{(i,j)}{m}{}^{p_1\ \ldots\ p_n}_{u_1 \ldots u_n}}{m^{(i,j)}+n^{(i,j)}} \text{ and } \frac{\overset{(i,j)}{n}{}^{p_1\ \ldots\ p_n}_{u_1 \ldots u_n}}{m^{(i,j)}+n^{(i,j)}} (m^{(i,j)}+n^{(i,j)} \neq 0;\ p_1=0, \ldots, l_1;\ \ldots;\ p_n=0, \ldots, l_n)$$

be seminvariants, where l_r *is the highest order of a derivative of* $[D^{(i,j)}]$ *(or* $\{D^{(i,j)}\}$*) with respect to* u_r $(r=1, \ldots, n)$ *occurring in the original seminvariant.*

6. *Application of the New Method to Special Cases.*

It should be noted that while we have indicated a large class of expressions which may be used in the place of $[D^{(i,j)}]$ or $\{D^{(i,j)}\}$, if we wish the simplest set of seminvariants we will set $m^{(i,j)}=1$, $n^{(i,j)}=0$, or $m^{(i,j)}=0$, $n^{(i,j)}=1$.

Let us now illustrate this method of computing seminvariants by the case of plane nets. We assume temporarily $a'_u=b'_v$, and take $\lambda_u/\lambda=b'$, $\lambda_v/\lambda=a'$. Substitution of these values in (3) yields the fundamental set of seminvariants

$$\left.\begin{array}{lll} A_1=a-2b', & B_1=b, & C_1=c+ab'+a'b-b'_u-b'^2,^* \\ A'_1=0, & B'_1=0, & C'_1=c'+a'b'-b'_v, \text{ or } c'+a'b'-a'_u, \\ A''_1=a'', & B''_1=b''-2a', & C''_1=c''+a''b'+a'b''-a'_v-a'^2,^* \end{array}\right\} \quad (12)$$

which are considerably simpler than those of (4). A further advantage of the new process over the old becomes evident when we discuss the invariants of (1). Of the seminvariants (4), only two, A'' and B, are invariants. Of the seminvariants (12), *five* are invariants, viz., those just mentioned and in addition $C_1=\mathfrak{C}$, $C'_1=K$ (or H), $C''_1=\mathfrak{C}''$. That this advantage can be guaranteed in general, however, is not evident. The simplicity of the new process might perhaps give it a better chance than the old to yield a larger number of seminvariants which are invariants as well. Another pseudo-canonical form, viz., that determined by $\lambda_u/\lambda=\frac{1}{2}a$, $\lambda_v/\lambda=\frac{1}{2}b''$, yields a fundamental set of seminvariants which are simpler than those of (4), but not quite so simple as those of (12). Only two of these are invariants.

In the case of conjugate systems of curves on a curved surface,† the new process (for a certain choice of λ) gives seminvariants which are very much simpler than those given by the regulation method. Moreover, three of the

* It happens that C_1 and C_1'' may be still further simplified by use of the integrability conditions.

† G. M. Green, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVII (1915), pp. 215–246.

five are seminvariants, as against one out of five for the old method. In the case of developable surfaces,* besides yielding simpler seminvariants, the new method gives two invariants, while the other gives none.

It seems evident that the method we have outlined will be of even greater value in the cases of higher dimensional configurations. For example, the simplification due to the new process for the case of curvilinear coordinates in n dimensions † is seen by noticing that it would replace, for every i,

$$f^{(i)}=\sum_{j=1}^{n} a_j^{(j,\,i)}$$

by some single coefficient $a_j^{(j,\,i)}$.

ANN ARBOR, MICH., *January* 2, 1918.

* W. W. Denton, *Trans. Amer. Math. Soc.*, Vol. XIV (1913), pp. 175–208.

† G. M. Green, "The Linear Dependence of Functions," etc. *loc. cit.* Cf. especially Section 7, and equations (21).

On a Method for Determining the Non-Stationary State of Heat in an Ellipsoid.

BY BIBHUTIBHUSAN DATTA.

Introduction.

1. The first writer, who attempted, with some success, the problem of the determination of the non-stationary state of heat in an *ellipsoid with three unequal axes*, was E. Mathieu,* who showed how the problem could be reduced to the solution of certain ordinary linear differential equations. But he found these equations to be so unmanageable that he contented himself with approximating to their solutions for the special case of an *ellipsoid of revolution*. Prof. C. Niven improved upon the results of Mathieu in certain respects in an interesting memoir,† entitled "On the Conduction of Heat in Ellipsoids of Revolution."

In the present paper, I propose (1) to obtain the chief results of Prof. Niven by an entirely different method, and (2), to show how this method can be applied to the case of the ellipsoid with three unequal axes, to obtain similar results which are believed to be new. It may be noted here that, in Art. 6, I point out a mistake in Prof. Niven's memoir.

Preliminary Remarks and Definitions.

2. Let the initial temperature of the ellipsoid be $f(x, y, z)$ and let its boundary, viz., $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, be maintained at temperature zero. Then the required temperature $V(x, y, z, t)$ is such that

$$\frac{\partial V}{\partial t}=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}, \tag{1}$$

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* *Cours de physique mathématique*, Ch. IX.

† *Phil. Trans.*, Vol. CLXXI (1880).

the units being so chosen that the diffusivity is 1,

$$V=0 \quad \text{on the boundary,} \tag{2}$$

$$V=f(x, y, z) \text{ when } t=0. \tag{3}$$

Thus V can be expressed as a sum of terms of the form $Ae^{-\lambda^2 t}W(x, y, z)$, where the normal function W satisfies the equation

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}=-\lambda^2 W,$$

and vanishes on the boundary; and the constants A are so chosen that the initial condition (3) is satisfied, so that $f(x, y, z)=\Sigma AW$.

When $a=b=c$ so that the ellipsoid becomes a sphere of radius a, an appropriate normal function is

$$S_n(\lambda r)P_n^m(\cos\theta)\cos m\phi,$$

where

$$S_n(x)=(-1)^n\sqrt{\frac{\pi}{2}}x^{-\frac{1}{2}}J_{n+\frac{1}{2}}(x), \quad P_n^m(\cos\theta)=\sin^m\theta\frac{d^m P_n}{d\mu^m}, \ \mu \text{ being } \cos\theta,$$

and λ is a root of the equation $S_n(\lambda a)=0$. Throughout the present paper I will represent by W_n^m the normal function corresponding to the function $S_n(\lambda r)P_n^m(\cos\theta)\cos m\phi$, and denote W_n^0 by W_n.

I proceed now to obtain the functions W of various types.

W_0 For Ellipsoid of Revolution.

3. Let e denote the eccentricity of the ellipsoid, then neglecting e^4 and higher powers, the equation of the ellipsoid can be written as

$$r=a\{1-\tfrac{1}{3}e^2+\tfrac{1}{3}e^2P_2(\cos\theta)\}, \quad i.\,e., \quad r=\alpha(1+\varepsilon P_2(\cos\theta)),$$

where $\alpha=a(1-\frac{1}{3}e^2)$ and $\varepsilon=\frac{1}{3}e^2$. Now assume that

$$W_0=S_0(\lambda r)+\varepsilon\sum_{t=1}^{\infty}M_tS_t(\lambda r)P_t(\cos\theta),$$

M_t being an unknown constant to be determined. Then, evidently, W_0 satisfies the partial differential equation (I); and to satisfy the boundary condition we must have

$$0=S_0(\lambda\alpha)+\varepsilon\alpha\frac{\partial S_0(\lambda\alpha)}{\partial\alpha}P_2+\varepsilon\sum_{t=1}^{\infty}M_tS_t(\lambda\alpha)P_t(\cos\theta),$$

since ε^2 and higher powers are neglected. This equation must hold for all

values of $\cos\theta$. Therefore, equating to zero the coefficients of the various zonal harmonics, we get

$$S_0(\lambda\alpha)=0, \tag{1}$$

$$M_2S_2(\lambda\alpha)+\alpha\frac{\partial S_0(\lambda\alpha)}{\partial\alpha}=0, \tag{2}$$

and all the other M's are zero. Hence the required expression for W_0, in terms of a and e, is

$$W_0=S_0(\lambda r)-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}S_2(\lambda r)P_2(\cos\theta),$$

where λ is given by the equation (1). But the general solution of the equation (1) is known to be

$$\lambda\alpha=i\pi,$$

i being any integer. Hence

$$\lambda a=i\pi+\tfrac{1}{3}i\pi e^2.$$

4. In order to obtain a closer approximation to W_0, we will retain e^4 and neglect the sixth and higher powers. Thus the equation of the ellipsoid is

$$r=a[(1-\tfrac{1}{3}e^2-\tfrac{2}{15}e^4)+(\tfrac{1}{3}e^2+\tfrac{1}{21}e^4)P_2+\tfrac{8}{35}e^4P_4],\ \textit{i. e.},\ r=\beta[1+\sigma P_2+\tau P_4],$$

where $\beta=a(1-\frac{1}{3}e^2-\frac{2}{15}e^4)$, $\sigma=\frac{1}{3}e^2+\frac{10}{63}e^4$, and $\tau=\frac{8}{35}e^4$.

Let us assume that

$$W_0=S_0(\lambda r)-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}S_2(\lambda r)P_2(\cos\theta)+\tau\sum_{t=1}^{\infty}N_tS_t(\lambda r)P_t(\cos\theta),$$

N_t being an unknown constant to be determined. Expanding by Taylor's Theorem, we have

$$S_0(\lambda r)=S_0(\lambda\beta)+\beta\frac{\partial S_0(\lambda\beta)}{\partial\beta}(\sigma P_2+\tau P_4)+\tfrac{1}{2!}\beta^2\frac{\partial^2S_0(\lambda\beta)}{\partial\beta^2}(\sigma P_2+\tau P_4)^2+\ldots,$$

and
$$S_2(\lambda r)=S_2(\lambda\beta)+\beta\frac{\partial S_2(\lambda\beta)}{\partial\beta}(\sigma P_2+\tau P_4)+\ldots,$$

when
$$r=\beta(1+\sigma P_2+\tau P_4).$$

Again
$$(P_2)^2=\tfrac{18}{35}P_4+\tfrac{2}{7}P_2+\tfrac{1}{5}.$$

Hence we must have

$$0=S_0(\lambda\beta)+\beta\frac{\partial S_0(\lambda\beta)}{\partial\beta}(\sigma P_2+\tau P_4)+\frac{\sigma^2}{2}\beta^2\frac{\partial^2 S_0(\lambda\beta)}{\partial\beta^2}[\tfrac{18}{35}P_4+\tfrac{2}{7}P_2+\tfrac{1}{5}]$$

$$-\tfrac{1}{8}e^2\frac{a\frac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}S_2(\lambda\beta)P_2-\tfrac{1}{8}e^2\sigma\frac{a\frac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\beta\frac{\partial S_2(\lambda\beta)}{\partial\beta}[\tfrac{18}{35}P_4+\tfrac{2}{7}P_2+\tfrac{1}{5}]$$

$$+\tau\sum_{i=1}^{\infty}N_iS_i(\lambda\beta)P_i.$$

This equation must hold for all values of $\cos\theta$. Therefore, equating to zero the coefficients of the various zonal harmonics, we get

$$S_0(\lambda\beta)+\tfrac{1}{10}\sigma^2\beta^2\frac{\partial^2 S_0(\lambda\beta)}{\partial\beta^2}-\tfrac{1}{15}e^2\sigma\frac{a\frac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot\beta\frac{\partial S_2(\lambda\beta)}{\partial\beta}=0, \tag{1}$$

$$\tau N_2S_2(\lambda\beta)+\sigma\beta\frac{\partial S_0(\lambda\beta)}{\partial\beta}+\tfrac{1}{7}\sigma^2\beta^2\frac{\partial^2 S_0(\lambda\beta)}{\partial\beta^2}-\tfrac{1}{8}e^2\frac{a\frac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot S_2(\lambda\beta)$$
$$-\tfrac{2}{21}e^2\sigma\frac{a\frac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot\beta\frac{\partial S_2(\lambda\beta)}{\partial\beta}=0, \tag{2}$$

$$\tau N_4S_4(\lambda\beta)+\tau\beta\frac{\partial S_0(\lambda\beta)}{\partial\beta}+\tfrac{9}{35}\sigma^2\beta^2\frac{\partial^2 S_0(\lambda\beta)}{\partial\beta^2}$$
$$-\tfrac{6}{35}e^2\sigma\frac{a\frac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot\beta\frac{\partial S_2(\lambda\beta)}{\partial\beta}=0; \tag{3}$$

and all the other N's are zero. On substituting the values of β, σ and τ in terms of a and e, the above equations take the forms

$$S_0(\lambda a)-(\tfrac{1}{8}e^2+\tfrac{2}{15}e^4)a\frac{\partial S_0(\lambda a)}{\partial a}+\tfrac{1}{15}e^4a^2\frac{\partial^2 S_0(\lambda a)}{\partial a^2}$$
$$-\tfrac{1}{45}e^4\frac{a\frac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\,a\frac{\partial S_2(\lambda a)}{\partial a}=0, \tag{4}$$

$$\tfrac{8}{35}N_2S_2(\lambda a)+\tfrac{1}{21}a\frac{\partial S_0(\lambda a)}{\partial a}-\tfrac{2}{21}a^2\frac{\partial^2 S_0(\lambda a)}{\partial a^2}+\tfrac{5}{63}\cdot\frac{a\frac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot a\frac{\partial S_2(\lambda a)}{\partial a}=0, \tag{5}$$

$$\tfrac{8}{35}N_4S_4(\lambda a)+\tfrac{8}{35}a\frac{\partial S_0(\lambda a)}{\partial a}+\tfrac{1}{35}a^2\frac{\partial^2 S_0(\lambda a)}{\partial a^2}-\tfrac{2}{35}\cdot\frac{a\frac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot a\frac{\partial S_2(\lambda a)}{\partial a}=0. \tag{6}$$

Hence we must have

$$0=S_n(\lambda\alpha)P_n+\varepsilon\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}[B_{n+2}P_{n+2}+C_nP_n+D_{n-2}P_{n-2}]+\varepsilon\sum_{t=0}^{\infty}{}' H_tS_t(\lambda\alpha)P_t.$$

This must be true for all values of $\cos\theta$. Therefore equating to zero the coefficients of the various zonal harmonics, we get

$$S_n(\lambda\alpha)+\varepsilon\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}C_n=0, \tag{1}$$

$$H_{n+2}S_{n+2}(\lambda\alpha)+\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}B_{n+2}=0, \tag{2}$$

$$H_{n-2}S_{n-2}(\lambda\alpha)+\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}D_{n-2}=0, \tag{3}$$

and all the other H's are zero. Thus the unknown constants are determined and the required expression for W_n, in terms of a and e, is

$$W_n=S_n(\lambda r)P_n-\tfrac{1}{8}e^2\frac{a\dfrac{\partial S_n(\lambda a)}{\partial a}}{S_{n+2}(\lambda a)}S_{n+2}(\lambda r)P_{n+2}B_{n+2}$$
$$-\tfrac{1}{8}e^2\frac{a\dfrac{\partial S_n(\lambda a)}{\partial a}}{S_{n-2}(\lambda a)}S_{n-2}(\lambda r)P_{n-2}D_{n-2};$$

λ being a root of the equation (1).

But, in terms of a and e, this equation is

$$S_n(\lambda a)-\frac{n^2+n-1}{(2n+3)(2n-1)}e^2a\frac{\partial S_n(\lambda a)}{\partial a}=0. \tag{4}$$

Therefore, if x be a root of the equation $S_n(x)=0$, the corresponding solution of (4) is given by

$$\lambda a=x\left\{1+\frac{n^2+n-1}{(2n+3)(2n-1)}e^2\right\}.$$

W_n^m For Ellipsoid of Revolution.

8. Let e^4 and higher powers be neglected, and assume that

$$W_n^m=S_n(\lambda r)P_n^m(\cos\theta)\cos m\phi+\varepsilon\sum_{t=m}^{\infty}{}' I_t^mS_t(\lambda r)P_t^m(\cos\theta)\cos m\phi,$$

where $\sum_{t=m}^{\infty}{}'$ refers to all values of t from m up to infinity except the value $t=n$, and I_t^m is an unknown constant to be determined. Then it is evident that W_n^m

satisfies the partial differential equation (I). Now, putting $r=a(1+\varepsilon P_2)$, we get

$$S_n(\lambda r)=S_n(\lambda a)+\varepsilon a\frac{\partial S_n(\lambda a)}{\partial a}P_2;\ \text{and}\ P_2\cdot P_n^m=B_{n+2}^m P_{n+2}^m+C_n^m P_n^m+D_{n-2}^m P_{n-2}^m,$$

where

$$B_{n+2}^m=\tfrac{3}{2}\cdot\frac{(n-m+1)(n-m+2)}{(2n+3)(2n+1)},\quad C_n^m=\frac{n(n+1)-3m^2}{(2n+3)(2n-1)},$$

$$D_{n-2}^m=\tfrac{3}{2}\cdot\frac{(n+m)(n+m-1)}{(2n+1)(2n-1)}.$$

Hence, from the boundary condition, we must have

$$0=S_n(\lambda a)P_n^m\cos m\phi+\varepsilon a\frac{\partial S_n(\lambda a)}{\partial a}[B_{n+2}^m P_{n+2}^m+C_n^m P_n^m+D_{n-2}^m P_{n-2}^m]$$
$$\times\cos m\phi+\varepsilon\sum_{t=m}^{\infty}{}' I_t^m S_t(\lambda a)P_t^m\cos m\phi.$$

Therefore, equating to zero the coefficients of the various surface harmonics, we get

$$S_n(\lambda a)+\varepsilon a\frac{\partial S_n(\lambda a)}{\partial a}C_n^m=0, \tag{1}$$

$$I_{n+2}^m S_{n+2}(\lambda a)+a\frac{\partial S_n(\lambda a)}{\partial a}B_{n+2}^m=0, \tag{2}$$

$$I_{n-2}^m S_{n-2}(\lambda a)+a\frac{\partial S_n(\lambda a)}{\partial a}D_{n-2}^m=0, \tag{3}$$

and all the other I_t^m's are zero.

Thus the required expression for W_n^m in terms of a and e is

$$W_n^m=\left[S_n(\lambda r)P_n^m(\cos\theta)-\tfrac{1}{3}e^2\frac{a\frac{\partial S_n(\lambda a)}{\partial a}}{S_{n+2}(\lambda a)}S_{n+2}(\lambda r)P_{n+2}^m(\cos\theta)B_{n+2}^m\right.$$
$$\left.-\tfrac{1}{3}e^2\frac{a\frac{\partial S_n(\lambda a)}{\partial a}}{S_{n-2}(\lambda a)}S_{n-2}(\lambda r)P_{n-2}^m(\cos\theta)D_{n-2}^m\right]\cos m\phi,$$

where λ is a root of the equation (1). But, expressed in terms of a and e, this equation becomes

$$S_n(\lambda a)-\frac{(n^2+n-1)+m^2}{(2n+3)(2n-1)}e^2 a\frac{\partial S_n(\lambda a)}{\partial a}=0. \tag{4}$$

Therefore, we obtain

$$\lambda a=x\left\{1+\frac{(n^2+n-1)+m^2}{(2n+3)(2n-1)}e^2\right\},$$

corresponding to the root x of $S_n(x)=0$.

W_0 *For Ellipsoid with Three Unequal Axes.*

9. Let e_1 and e_2 be the eccentricities of the two principal diametral sections of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $a>b>c$, by planes passing through the major axis. Then, neglecting the fourth and higher powers of e_1 and e_2, the equation to the ellipsoid is

$$r=a[1-\tfrac{1}{6}(e_1^2+e_2^2)(P_0-P_2)+\tfrac{1}{12}(e_2^2-e_1^2)P_2^2\cos 2\phi],$$

i. e.,
$$r=\gamma(1+\varepsilon_1 P_2+\varepsilon_2 P_2^2\cos 2\phi),$$

where $\gamma=a\{1-\frac{1}{6}(e_1^2+e_2^2)\}$, $\varepsilon_1=\frac{1}{6}(e_1^2+e_2^2)$, and $\varepsilon_2=\frac{1}{12}(e_2^2-e_1^2)$.

Now assume that

$$W_0=S_0(\lambda r)+\sum_{t=m}^{\infty}{}' \sum_{m=0}^{\infty} R_{m,t}S_t(\lambda r)P_t^m(\cos\theta)\cos m\phi,$$

where $R_{m,t}$ is an unknown constant to be determined and $\sum_{t=m}^{\infty}{}'$ refers to all values of t from m up to ∞, except $t=0$. Then it is evident that W_0 satisfies the partial differential equation (I). To satisfy the boundary condition, we must have

$$0=S_0(\lambda\gamma)+\gamma\frac{\partial S_0(\lambda\gamma)}{\partial\gamma}(\varepsilon_1 P_2+\varepsilon_2 P_2^2\cos 2\phi)+\sum_{t=m}^{\infty}{}' \sum_{m=0}^{\infty} R_{m,t}S_t(\lambda\gamma)P_t^m(\cos\theta)\cos m\phi,$$

the unknown constants R's being assumed to be of the same order as ε_1 and ε_2. Therefore, equating to zero the coefficients of the various surface harmonics, we get

$$S_0(\lambda\gamma)=0, \tag{1}$$

$$R_{0,2}S_2(\lambda\gamma)+\varepsilon_1\gamma\frac{\partial S_0(\lambda\gamma)}{\partial\gamma}=0, \tag{2}$$

$$R_{2,2}S_2(\lambda\gamma)+\varepsilon_2\gamma\frac{\partial S_0(\lambda\gamma)}{\partial\gamma}=0; \tag{3}$$

and all the other R's are zero. Thus the R's are determined, and we get finally the required expression for W_0 to be

$$W_0=S_0(\lambda r)-\tfrac{1}{6}(e_1^2+e_2^2)\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}S_2(\lambda r)P_2(\cos\theta)$$
$$+\tfrac{1}{12}(e_1^2-e_2^2)\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}S_2(\lambda r)P_2^2(\cos\theta)\cos 2\phi,$$

where λ is a root of the equation (1). But the roots of the equation (1) are given by

$$\lambda\gamma = i\pi.$$

Hence, in terms of a and the eccentricities, we have

$$\lambda a = i\pi + \tfrac{1}{6}(e_1^2 + e_2^2)\, i\pi.$$

Conclusion.

10. I conclude this paper by pointing out that the results or Arts. 3–9 admit of extension and generalization in various directions. For example, a procedure similar to that of Art. 9 will give us W_n^m for the ellipsoid with three unequal axes. Also, denoting by $\overline{W}_n^m$, the normal function corresponding to

$$S_n(\lambda r) P_n^m(\cos\theta) \sin m\phi,$$

it is obvious that, for the ellipsoid of revolution as well as for the ellipsoid with three unequal axes, $\overline{W}_n^m$ can be obtained in the same way as W_n^m.

UNIVERSITY OF CALCUTTA, 1917.

Nilpotent Algebras Generated by Two Units, i and j, Such That i^2 Is Not an Independent Unit.*

BY GUY WATSON SMITH.

I. *Introduction.*

The problem of referring all hypercomplex number systems to a relatively small number of typical forms was suggested by Hamilton,† but with the exception of De Morgan's discussion of double and triple algebras, nothing much was done till Benjamin Peirce ‡ worked out all algebras of deficiency zero and one. Starkweather § worked out algebras of deficiency two. He showed also that algebras of n units could be obtained from those of $n-1$ units. Cartan ‖ using the characteristic equation developed the semi-simple and the nilpotent sub-algebras, and showed the possibility of representing every algebra by means of units with double character. Taber ¶ reestablished the results of Peirce and extended them to any domain of rationality for the coordinates. Wedderburn ** and Voghera †† made an advance in the treatment of the hypercomplex algebra by basing their work on the conception of invariant classes of numbers in the algebra.

Besides this direct line of development there have been two others. The first is by means of the continuous group, the second is by using the matrix

* This paper considers only linear associative algebras whose coordinates are taken from the field of ordinary complex numbers.

† "Lectures on Quaternions," Preface, pp, 29–31.

‡ "Linear Associative Algebra," AMERICAN JOURNAL OF MATHEMATICS, Vol. IV (1881) pp. 97–192.

§ "Non-Quaternion Number Systems Containing no Skew Units," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), pp. 369–386.

‖ "Les groupes bilineaires et les systemes de nombres complexes, *Annales de la Faculte des Sciences de Toulouse*, Vol. XII (1898), B. pp. 1–99.

¶ "On Hypercomplex Number Systems," *Am. Math. Soc. Trans.*, Vol. V (1904), pp. 509–548.

** "On Hypercomplex Numbers," *London Math. Soc. Proceedings*, Series 2, Vol. VI (1907), pp. 77–118.

†† "Zusammenstellung der Irreduziblen Komplexen Zahlensysteme in sechs Einheiten," *Denkschriften der Math. Nat. Klasse der Kais. Akad. der Wiss.* Wien., Vol. LXXXIV (1908), pp. 269–329.

theory. The first method was used by Scheffers,* Molien,† and Study,‡ the second by Shaw§ and Frobenius.||

Shaw regards all associative numbers as belonging to an associative algebra of an infinite number of units, the "associative units," λ_{rst}, which are elementary matrices. Each associative number is a linear combination of these units, so the theory of linear associative algebra is the theory of these associative units. He shows that the presence of a modulus is not necessary, thus making the methods particularly applicable to nilpotent systems. He proves that the equation of an algebra determines all the units but those which form a nilpotent system, and consequently to get all linear associative algebras we must first determine all nilpotent algebras. Benjamin Peirce¶ was the first to recognize the importance of nilpotent algebras. Furthermore, algebras of order n may be found without first knowing those of order $n-1$. By selecting a base and adjoining to this a nilpotent unit, an ever-increasing system of nilpotent algebras may be determined.

The simplest case of the Shaw canonical form** of a nilpotent algebra is that in which there is only a single generator, j, whose μ_1-th power vanishes,

$$j,\ j^2,\ j^3,\ j^4,\ \ldots,\ j^{\mu_1-1}.$$

Three lemmas concerning polynomials in this nilpotent have been introduced for use in the handling of the next simplest case, namely, that in which there are two generators, j and another nilpotent unit, i, whose square is not an independent unit in the canonical form. The unit j is such that $j^{\mu_1-1}\neq 0$, $j^{\mu_1}=0$ and $ij^{\mu_2-1}\neq 0$, $ij^{\mu_2}=0$, where μ_1 and μ_2 are multiplicities of j relative to j and to i respectively, and $\mu_1 \geqq \mu_2$. The expressions i, j^s, ij^t where $0<s<\mu_1$, and $0<t<\mu_2$, are the independent units of the system. The algebra is therefore of order $\mu_1+\mu_2-1$, or if we insert a modulus η, of order $\mu_1+\mu_2$. The μ_1 power of every number must vanish, hence the deficiency in this case is μ_2. There is at least one hypernumber which does not vanish for a power lower than μ_1,

* "Zurückführung complexer Zahlensysteme auf typische Formen," *Math. Annalen*, Vol. XXXIX (1891), pp. 293–390.

† "Ueber Systeme höherer complexen Zahlen," *Math. Annalen*, Vol. XLI (1893) pp. 83–156.

‡ "Ueber Systeme von complexen Zahlen," *Gött. Nach.* (1889), pp. 237–268. "Complexe Zahlen und Transformationsgruppen," *Leipzig Berichte*, Vol. XLI (1889), pp. 177–228.

§ "Theory of Linear Associative Algebra," *Am. Math. Soc. Trans.*, Vol. IV (1903) pp. 251–287. "On Nilpotent Algebras," *Am. Math. Soc. Trans.*, Vol. IV (1903), pp. 405–422.

|| "Theorie der hypercomplexen Grossen," *Berlin Berichte* (1903), pp. 504–537, 634–645.

¶ *Loc. cit.*, p. 118.

** *Loc. cit.*, p. 406.

and there may be several such. The base unit i is chosen such that $ij^{\mu_2}=0$, $ij^{\mu_2-1}\neq 0$, the various products ij, ij^2, ..., do not contain terms in j alone, such as aj^m, and μ_2 is such that there is no hypernumber i' satisfying the conditions on i for which $i'j^{\mu_3}=0$, $i'j^{\mu_3-1}\neq 0$, $\mu_3>\mu_2$. In other words, i is not the product of any hypernumber into j, and we do not have for any power of j, $ij^s=bj^t$. These characters of i and j are essential. The two sequences $j, j^2, \ldots, j^{\mu_1-1}$ and $i, ij, ij^2, \ldots, ij^{\mu_2-1}$ constitute the two shear regions of j. In the cases Peirce considered, that of deficiency zero contains no i; that of deficiency unity must have $\mu_2=1$, hence $ij=0$. In Starkweather's types where only two generators enter, $\mu_2=2$, hence ij is a unit but $ij^2=0$. In the types herein considered the deficiency does not play any rôle at all. The investigation is along the line of Shaw's construction by *generators* and not by classification by numerical invariants, other than those entering the equations of condition. The associative units are used, though not indispensable, for convenience in making reductions.

II. *Lemmas Concerning Polynomials of a Single Nilpotent j.*

LEMMA I. *If j is a nilpotent number ($j^{\mu_1}=0$), and if $F(j)$ and $G(j)$ are polynomials in j, a quotient $Q(j)$ can always be obtained $\dfrac{F(j)}{G(j)}$ provided $F(j)$ does not contain a term of lower order than does $G(j)$.*

PROOF: The lemma will be proved if we can find a polynomial, $Q(j)$ of not more than μ_1 terms such that $F(j)=Q(j)\cdot G(j)$, or

$$f_0+f_1j+f_2j^2+\ldots\ldots=(q_0+q_1j+q_2j^2+\ldots\ldots)(g_0+g_1j+g_2j^2+\ldots\ldots).$$

Equating coefficients of corresponding powers of j on both sides since they belong to independent units,

$$f_0=q_0g_0,\quad f_1=q_0g_1+q_1g_0,\quad f_2=q_0g_2+q_1g_1+q_2g_0,\quad \ldots\ldots\ldots\ldots$$

These equations can not be solved if $F(j)$ starts with a lower power of j than $G(j)$, that is, if

$$f_0=f_1=f_2=\ldots\ldots=f_h=0,\ f_{h+1}\neq 0,$$

while $g_0=g_1=\ldots\ldots=g_{k+h}=0$ $k>0$. If, however, $f_0=f_1=\ldots\ldots=f_h=0$, $f_{h+1}\neq 0$, while $g_0=g_1=\ldots\ldots=g_{h-k}=0$, $g_{h-k+1}\neq 0$, $0<k<h+1$, then the first $h-k+1$ of the above equations are identically satisfied, the next k equations give $q_0=q_1=\ldots\ldots=q_{k-1}=0$ and from the remaining μ_1-h-1 equations, q_k, $q_{k+1}, \ldots, q_{\mu_1-h+k-1}$ may be uniquely determined, leaving $q_{\mu_1-h+k}, q_{\mu_1-h+k+1}, \ldots,$ q_{μ_1-1} arbitrary.

LEMMA II. *Consider the equation* $L(j)\cdot M(j)=0$ *where*

$$L(j)=l_0+l_1j+l_2j^2+\ldots\ldots+l_{\mu_1-1}j^{\mu_1-1},\quad M(j)=m_0+m_1j+m_2j^2+\ldots\ldots+m_{\mu_1-1}j^{\mu_1-1}.$$

(a). If $l_0\neq 0$, $M(j)$ vanishes entirely.

PROOF: $(l_0+l_1j+l_2j^2+\ldots\ldots+l_{\mu_1-1}j^{\mu_1-1})(m_0+m_1j+\ldots\ldots+m_{\mu_1-1}j^{\mu_1-1})=0$. Then

$$l_0m_0=0\ \therefore\ m_0=0,\quad l_0m_1+l_1m_0=0\ \therefore\ m_1=0,\quad l_0m_2+l_1m_1+l_2m_2=0\ \therefore\ m_2=0,$$
$$\ldots\ldots\ldots\ldots\ldots,\quad l_0m_{\mu_1-1}+\text{ vanishing terms }=0\ \therefore\ m_{\mu_1-1}=0.$$

(b). If $l_0=l_1=l_2=\ldots\ldots=l_n=0$, $l_{n+1}\neq 0$, $n<\mu_1-1$, then $L=j^{n+1}L_{n+1}(j)$, $M(j)j^{n+1}L_{n+1}(j)=0$, where $L_{n+1}(j)=l_{n+1}+l_{n+2}j+\ldots\ldots+l_{\mu_1-1}j^{\mu_1-n-2}$ and $l_{n+1}\neq 0$. Applying (a) to the equation $M(j)j^{n+1}\cdot L_{n+1}(j)=0$ gives

$$M(j)j^{n+1}=m_0j^{n+1}+m_1j^{n+2}+\ldots\ldots+m_{\mu_1-n-2}j^{\mu_1-1}=0,$$
$$\therefore\ m_0=m_1=m_2=\ldots\ldots=m_{\mu_1-n-2}=0\ \text{ or }\ M(j)=j^{\mu_1-n-1}M_1(j),$$

where $M_1(j)=m_{10}+m_{11}j+m_{12}j^2+\ldots\ldots+m_{1n}j^n+$ any terms whatever up to j^{μ_1-1}.

LEMMA III. *If* $C(j)$ *is a polynomial in* j *with non-vanishing constant term, then* j *can be expressed as a polynomial in* (jC).

PROOF: Let $C(j)=a_0+a_1j+a_2j^2+\ldots\ldots+a_{\mu_1-1}j^{\mu_1-1}$, $a_0\neq 0$, then

$$jC=a_0j+a_1j^2+a_2j^3+\ldots\ldots+a_{\mu_1-2}j^{\mu_1-1}:$$

$$(jC)^2=a_0^2j^2+2a_0a_1j^3+(2a_0a_2+a_1^2)j^4+\ldots\ldots,\quad (jC)^3=a_0^3j^3+3a_0^2a_1j^4+\ldots\ldots,$$
$$(jC)^4=a_0^4j^4+\ldots\ldots,\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

And since $a_0, a_0^2, a_0^3, \ldots\ldots, a_0^{\mu_1-1}\neq 0$ by hypothesis the determinant of coefficients does not vanish, and we may solve these equations for $j, j^2, j^3, \ldots\ldots, j^{\mu_1-1}$ in terms of powers of (jC).

III. *Expressions for the Generators.*

From a nilpotent system choose any nilpotent expression j to be a unit, called the adjunct unit, such that j is a number which has as high a non-vanishing power as any number of the system, and a set of expressions (in this paper one, say i), called the base. Then it is known that any hypernumber of the system is linearly expressible in terms of the $\mu_1+\mu_2-1$ independent units

$$i,\ j,\ ij,\ j^2,\ ij^2,\ j^3,\ ij^3,\ \ldots\ldots,\ ij^{\mu_2-1},\ j^{\mu_2},\ j^{\mu_2+1},\ \ldots\ldots,\ j^{\mu_1-1},$$

and in this order the product of any two units ij^a and ij^b is linearly expressible in terms of the units which follow* the unit ij^{a+b-1}.

*Shaw, *Trans. Am. Math. Soc.*, Vol. IV (1903), pp. 406–410.

These types are now written out as follows:

11. $\mu_1>2\mu_2$, $\mu_1-\mu_2=\mu_2+\alpha$, $\alpha>0$, $E\neq 0$. The generators for this type are

$$i=210+12\delta^{\mu_2+\alpha}B,\quad j=111+221+12\delta^{\mu_2+\alpha}E.$$

From these we get the following products:

$$ij=211+12\delta^{\mu_2+\alpha+1}B,\quad j^2=112+222+12\delta^{\mu_2+\alpha+1}2E,$$
$$j^n=11n+22n+12\delta^{\mu_2+\alpha+n-1}nE,$$
$$ji=211+12\delta^{\mu_2+\alpha+1}B+11\delta^{\mu_2+\alpha}E=ij+j^{\mu_2+\alpha}E(j),\quad i^2=11\delta^{\mu_2+\alpha}B=j^{\mu_2+\alpha}B(j).$$

12. $\mu_1=2\mu_2$, $E=\delta E_1$, $E_1\neq 0$:

$$i=210+12\delta^{\mu_2}B,\quad j=111+221+12\delta^{\mu_2+1}E_1,\quad ij=211+12\delta^{\mu_2+1}B,$$
$$j^n=11n+22n+12\delta^{\mu_2+n}nE_1,$$
$$ji=211+12\delta^{\mu_2+1}B+11\delta^{\mu_2+1}E_1=ij+j^{\mu_2+1}E_1(j),\quad i^2=11\delta^{\mu_2}B=j^{\mu_2}B(j).$$

2. $\mu_1-\mu_2<\mu_2$, $E=\delta^v E_3$, $E_3\neq 0$, $v>2\mu_2-\mu_1$:

$$i=210+12\delta^{\mu_1-\mu_2}B,\quad j=111+221+12\delta^{\mu_1-\mu_2+v}E_3,$$
$$ij=211+12\delta^{\mu_1-\mu_2+1}B,\quad j^n=11n+22n+12\delta^{\mu_1-\mu_2+v+n-1}nE_3,$$
$$ji=211+12\delta^{\mu_1-\mu_2+1}B+11\delta^{\mu_1-\mu_2+v}E_3=ij+j^{\mu_1-\mu_2+v}E_3(j),$$
$$i^2=22\theta^{\mu_1-\mu_2}B+11\delta^{\mu_1-\mu_2}B=j^{\mu_1-\mu_2}B(j).$$

3. $E=0$:

$$i=210+12\delta^{\mu_1-\mu_2}B,\quad j=111+221,\quad ij=211+12\delta^{\mu_1-\mu_2+1}B,$$
$$ji=211+12\delta^{\mu_1-\mu_2+1}B=ij,\quad j^n=11n+22n,$$
$$i^2=11\delta^{\mu_1-\mu_2}B+22\theta^{\mu_1-\mu_2}B=j^{\mu_1-\mu_2}B(j).$$

Type	$ji'=$	$(i')^2=$
1	$i'j$,	$j^{\mu_1-\mu_2}B(j)$,
2	$i'j+j^{\mu_2+1}E_1(j)$,	$j^{\mu_2}B(j)$,
3	$i'j+j^{\mu_2+\alpha}E(j)$,	$j^{\mu_2+\alpha}B(j)$,
4	$i'j+j^{\mu_1-\mu_2+v}E_3(j)$,	$j^{\mu_1-\mu_2}B(j)$,

or in terms of i and j, instead of i' and j.

Type	$ji=$	$i^2=$
1	ij,	$ijA+\frac{1}{2}j^{\mu_1-\mu_2}E(j)(jA)'-\frac{1}{4}j^2A^2+j^{\mu_1-\mu_2}B(j)$,
2	$ij+j^{\mu_2+1}E_1(j)$,	$ijA+\frac{1}{2}j^{\mu_1-\mu_2}E(j)(jA)'-\frac{1}{4}j^2A^2+j^{\mu_2}B(j)$,
3	$ij+j^{\mu_2+\alpha}E(j)$,	$ijA+\frac{1}{2}j^{\mu_1-\mu_2}E(j)(jA)'-\frac{1}{4}j^2A^2+j^{\mu_1-\mu_2}B(j)$,
4	$ij+j^{\mu_1-\mu_2+v}E_3(j)$,	$ijA+\frac{1}{2}j^{\mu_1-\mu_2}E(j)(jA)'-\frac{1}{4}j^2A^2+j^{\mu_1-\mu_2}B(j)$,

when $(jA)'$ is the formal derivative as to j of the polynomial jA, and A is arbitrary. In case A starts with a term of order $\mu_1-\mu_2-1$ or higher, i' may be used for i.

V. *Case II.* $A=0,\ C\neq 1,\ E=0.$

If A vanishes we have the defining equations:

$$i=210+12\delta^{\mu_1-\mu_2}B, \quad j=111+22\theta C.$$

Then $\quad ij^s=21s+12\delta^{\mu_1-\mu_2+s}BC^s$, and $iP(j)=21P(\theta)+12\delta^{\mu_1-\mu_2}BP(\delta C)$.

Again $\quad ji=21\theta C+12\delta^{\mu_1-\mu_2+1}B=ijC(j)-12\delta^{\mu_1-\mu_2+1}BC\cdot C(\delta C)+12\delta^{\mu_1-\mu_2+1}B$

$$\therefore\ \delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0. \tag{1}$$

Also $\quad j^2=112+22\theta^2C^2,\ \ldots\ldots\ldots\ldots\ldots,$ whence $P(j)=11P+22P(\theta C)$,

$\quad i^2=11\delta^{\mu_1-\mu_2}B+22\theta^{\mu_1-\mu_2}B,$

but $\quad j^{\mu_1-\mu_2}=11\delta^{\mu_1-\mu_2}+22\theta^{\mu_1-\mu_2}C^{\mu_1-\mu_2}$, so $i^2=j^{\mu_1-\mu_2}B(j)$,

$$\therefore\ \theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0. \tag{2}$$

Let C be written $C=c_0+\theta^tC_1$ where c_0 may be 0, but $c_{10}\neq 0$ and $0<t<\mu_2-1$. Substituting in equation (1)

$$\delta^{\mu_1-\mu_2+1}B[1-(c_0+\theta^tC_1)(c_0+\theta^tC^tC_1(\theta C))]=0,$$

or

$$\delta^{\mu_1-\mu_2+1}B[1-c_0^2-\theta^tc_0C_1-c_0\theta^tC^tC_1(\theta C)]=0.$$

By applying Lemma II we have either

11. $c_0^2\neq 1$ and $B=\delta^{\mu_2-1}b_{\mu_2-1}$,

or 12. $c_0^2=1$ and $B=\delta^{\mu_2-t-1}B_1$.

Now substituting in equation (2) the results just found, we have for 11:

$$\theta^{\mu_1-1}b_{\mu_2-1}[1-(c_0+\theta^tC_1)^{\mu_1-1}]=0.$$

Therefore by Lemma II,

111, $b_{\mu_2-1}=0$, and $B=0$; or 112, $c_0^{\mu_1-1}=1$, $b_{\mu_2-1}\neq 0$.

For the case 12, $\quad \theta^{\mu_1-t-1}[B_1-C^{\mu_1-t-1}B_1(\theta C)]=0.$

Hence either 121, for $c_0=1$, $\theta^{\mu_1-1}b_{10}(\mu_1-t-1)C_1=0$, for which we have either

1211, $\mu_1>\mu_2$, $B=\theta^{\mu_2-t-1}B_1$,

or 1212, $\mu_1=\mu_2$, $B=\theta^{\mu_2-t}B_2$,

or 122, for $c_0=-1$, $\theta^{\mu_1-t-1}[B_1-(-1+\theta^tC_1)^{\mu_1-t-1}B_1(\theta C)]=0$,

for which we have

1221, μ_1-1 is even, $B_1=\theta^tB_3$, *i. e.*, $B=\theta^{\mu_2-1}b_{\mu_2-1}$, $b_{\mu_2-1}\neq 0$,

or 1222, $B_1=\theta^{t+1}B_4$, *i. e.*, $B=0$,

or 1223, $\mu_1-t-1\geq\mu_2$, $B_1\neq 0$.

Consequently when $A=0$ the following seven types arise:

$$A=0, \quad C=c_0+\theta^t C_1,$$

111. $c_0^2 \neq 1$, $B=0$.
112. $c_0^{\mu_1-t}=1$, $B=\delta^{\mu_2-1}b_{\mu_2-1}$.
1211. $c_0=1$, $B=\delta^{\mu_1-t-1}B_1$, $\mu_1>\mu_2$.
1212 $c_0=1$, $B=\delta^{\mu_1-t}B_2$, $\mu_1=\mu_2$.
1221. $c_0=-1$, μ_1-1 is even, $B=\delta^{\mu_2-1}b_{\mu_2-1}$.
1222. $c_0=-1$, $B=0$.
1223 $c_0=-1$, $\mu_1-t-1 \geq \mu_2$, $B=\delta^{\mu_1-t-1}B_1$, $B_1 \neq 0$

We now work out these seven types obtaining the expressions for the products ji and i^2.

111. $C=c_0+\theta^t C_1$, $c_0^2 \neq 1$, $B=0$. For this the generators have the form
$i=210$, $j=111+22\theta C$.

Then $ij=211$, $ji=21\theta C=ijC(j)$, and $i^2=0$.

112. $c_0^{\mu_1-t}=1$, $B=\delta^{\mu_2-1}b_{\mu_2-1}$.

Now $i=210+12\delta^{\mu_2-1}b_{\mu_2-1}$, $j=111+22\theta C$, $ij=211$, $ji=21\theta C=ijC(j)$,
$i^2=11\delta^{\mu_1-1}b_{\mu_1-1}+22\theta^{\mu_1-1}b_{\mu_2-1}$, $j^n=11\delta^n+22\theta^n C^n$, $\therefore$ $i^2=j^{\mu_1-1}b_{\mu_1-1}$.

1211. $c_0=1$, $B=\delta^{\mu_1-t-1}B_1$, $\mu_1>\mu_2$.
$i=210+12\delta^{\mu_1-t-1}B_1$, $j=111+221+22\theta^{t+1}C_1$, $ij^n=21n+12\delta^{\mu_1-t+n-1}B_1$,
$ji=211+12\delta^{\mu_1-t}B_1+21\theta^{t+1}C_1=ij+ij^{t+1}C_1(j)$,
$j^n=11n+22\theta^n C^n=11n+22\theta^n(1+\theta^t C_1)^n$,
$i^2=11\delta^{\mu_1-t-1}B_1+22\theta^{\mu_1-t-1}B_1=j^{\mu_1-t-1}B_1(j)$.

1212. $c_0=1$, $B=\delta^{\mu_1-t}B_2$, $\mu_1=\mu_2$.
$i=210+12\delta^{\mu_1-t}B_2$, $j=111+221+22\theta^{t+1}C_1=111+22\theta C$,
$ij^n=21n+12\delta^{\mu_1-t+n}B_2$, $ji=211+21\theta^{t+1}C_1+12\delta^{\mu_1-t+1}B_2=ij+ij^{t+1}C_1(j)$,
$i^2=11\delta^{\mu_1-t}B_2+22\theta^{\mu_1-t}B_2=j^{\mu_1-t}B_2(j)$.

1221. μ_1-1 is even, $B=\delta^{\mu_1-1}b$, $c_0=-1$.
$i=210+12\delta^{\mu_1-1}b$; $j=111+221(-1)+22\theta^{t+1}C_1$, $ij=211$,
$ji=211(-1)+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j)$,
$j^n=11n+22\theta^n(-1+\theta^t C_1)^n$, $i^2=11\delta^{\mu_1-1}b+22\theta^{\mu_1-1}b=j^{\mu_1-1}b$.

1222. $c_0=-1$, $B=0$.
$i=210$, $j=111+221(-1)+22\theta^{t+1}C_1$, $ij=211$,
$ji=211(-1)+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j)$, $i^2=0$.

1223. $c_0=-1,\quad \mu_1-t-1\geq\mu_2,\quad B=\delta^{\mu_1-t-1}B_1,\quad B_1\neq 0.$

$$i=210+12\delta^{\mu_1-t-1}B_1,\quad j=111+221(-1)+22\theta^{t+1}C_1,$$
$$ij^n=21n+12\delta^{\mu_1-t+n-1}(-1)^nB_1,$$
$$ji=211(-1)+21\theta^{t+1}C_1+12\delta^{\mu_1-t}B_1=-ij+ij^{t+1}C_1(j),$$
$$i^2=11\delta^{\mu_1-t-1}B_1=j^{\mu_1-t-1}B_1(j).$$

Type	$ji=$	$i^2=$	
1	$ij[c_0+j^tC_1(j)]$,	0,	$c_0^2\neq 1$,
2	$ij[c_0+j^tC_1(j)]$,	$j^{\mu_1-1}b_{\mu_2-1}$,	$c_0^2\neq 1,\quad b_{\mu_2-1}\neq 0$,
3	$ij+ij^{t+1}C_1(j)$,	$j^{\mu_1-t-1}B_1(j)$,	$\mu_1>\mu_2$,
4	$ij+ij^{t+1}C_1(j)$,	$j^{\mu_1-t}B_2(j)$,	$\mu_1=\mu_2$,
5	$-ij+ij^{t+1}C_1(j)$,	$j^{\mu_1-1}b$,	μ_1 is odd, $b\neq 0$,
6	$-ij+ij^{t+1}C_1(j)$,	0,	
7	$-ij+ij^{t+1}C_1(j)$,	$j^{\mu_1-t-1}B_1(j)$,	$\mu_1-t-1\geq\mu_2,\quad B_1\neq 0.$

VI. *Case III.* $A\neq 0,\quad C\neq 1,\quad E=0.$

In this case, which is the most general, we first form the following products from the generators:

$$i=210+22\theta A+12\delta^{\mu_1-\mu_2}B,\quad j=111+22\theta C,$$
$$ij^r=21r+22\theta^{r+1}AC^r+12\delta^{\mu_1-\mu_2+r}BC^r,\quad r<\mu_2-1,$$
$$ij^{\mu_2-1}=21\theta^{\mu_2-1}+12\delta^{\mu_1-1}BC^{\mu_2-1},$$
$$iP(j)=210P(\theta)+22\theta AP(\theta C)+12\delta^{\mu_1-\mu_2}BP(\delta C),\quad \text{where } P(j) \text{ is a}$$

polynomial in j.

$$j^s=11s+22\theta^sC^s,\quad s<\mu_2,\quad j^s=11\theta^s,\quad \mu_1>s\geq\mu_2,$$
$$P(j)=11P(\delta)+22P(\theta C),$$
$$\begin{aligned}ji&=21\theta C+22\theta^2AC+12\delta^{\mu_1-\mu_2+1}B\\&=ijC(j)+22\theta^2AC+12\delta^{\mu_1-\mu_2+1}B-22\theta^2AC\cdot C(\theta C)-12\delta^{\mu_1-\mu_2+1}BCC(\theta C),\end{aligned}$$
$$P(j)\cdot i\theta=210P(\theta C)+22\theta AP(\theta C)+12\delta^{\mu_1-\mu_2}BP(\delta)=i\cdot P(jC),$$
$$\begin{aligned}i^2&=21\theta A+22\theta^2A^2+22\theta^{\mu_1-\mu_2}B+11\delta^{\mu_1-\mu_2}B+12\delta^{\mu_1-\mu_2+1}AB\\&=ijA(j)+j^{\mu_1-\mu_2}B(j)+22\theta^2A^2+22\theta^{\mu_1-\mu_2}B+12\delta^{\mu_1-\mu_2+1}AB\\&\qquad-22\theta^2ACA(\theta C)-22\theta^{\mu_1-\mu_2}C^{\mu_1-\mu_2}B(\theta C)-12\delta^{\mu_1-\mu_2}BCA(\theta C).\end{aligned}$$

Since the terms in $22()$ and $12()$ must vanish in both the products ji and i^2, we have four equations:

$$\left.\begin{aligned}&(1)\quad \theta^2AC[1-C(\theta C)]=0,\\&(2)\quad \delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0,\\&(3)\quad \delta^{\mu_1-\mu_2+1}B[A-C\cdot A(\delta C)]=0,\\&(4)\quad \theta^2A[A-C\cdot A(\theta C)]+\theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0.\end{aligned}\right\}\qquad (A)$$

Before reducing these we need a second theorem which we proceed to deduce.

If $A(j)=j^sA_1(j)$, $0\leq s<\mu_1-1$, then $i^2=ij^{s+1}A_1(j)+j^{\mu_1-\mu_2}B(j)$, where $a_{10}\neq 0$. By Lemma III we may now express a polynomial in j in terms of one in jC if $C=c_0+jC_1$. Let $A_1(j)=A_1'(jC)$ and determine i' from $i=i'A_1'(j)$ where $a_{10}\neq 0$, then

$$i^2=i'A_1'i'A_1'=i'j^{s+1}A_1(j)\cdot A_1'(j)+j^{\mu_1-\mu_2}B(j).$$

But using the formula $P(j)i=iP(jC)$,

$$A_1'(j)\cdot i=iA_1'(jC)=iA_1(j),$$
$$A_1'(j)\cdot i'=iA_1(j)A_1'^{-1}(j)=i'A_1(j).$$

Thence $$i'A_1'i'A_1'=i'^2A_1A_1'=i'j^{s+1}A_1(j)A_1'(j)+j^{\mu_1-\mu_2}B(j).$$

That is $$i'^2=i'j^{s+1}+j^{\mu_1-\mu_2}B(j)A_1^{-1}(j)A_1'^{-1}(j)=i'j^{s+1}+j^{\mu_1-\mu_2}B'.$$

THEOREM II. *If $C(j)=c_0+jC_1(j)$, where $c_0\neq 0$ then $j=P(jC)$ and $A_1'(j)$ is known, and invertible. Hence we may choose i so that the term $ij^{s+1}A_1(j)$ becomes simply ij^{s+1}, that is, we may take $A_1=1$.*

As a consequence of Theorem II we have two sub-cases,

III_1. $A(j)=j^s$, $C(j)=c_0+jC_1(j)$, $c_0\neq 0$.

III_2. $C(j)=j^tC_1(j)$, $t>0$, $c_{10}\neq 0$.

Case III_1. The generators may now be written

$$i=210+22\theta^{s+1}+12\delta^{\mu_1-\mu_2}B,\quad j=111+22\theta C,$$

and we shall have

$$ji=ijC(j),\quad i^2=ij^{s+1}+j^{\mu_1-\mu_2}B(j),$$

where the following four equations, which are the reduced forms of equations (A), must be simultaneously satisfied:

$$\left.\begin{array}{ll}(1)' & \theta^{s+2}C[1-C(\theta C)]=0,\\ (2)' & \delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0,\\ (3)' & \delta^{\mu_1-\mu_2+s+1}[1-C^{s+1}]=0,\\ (4)' & \theta^{2s+2}[1-C^{s+1}]+\theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0.\end{array}\right\}\qquad (A)'$$

Consider equation (1)$'$,

$$\theta^{s+2}C[1-C(\theta C)]=0.$$

This leads to two classes 3 and 4.

3 $s+2=\mu_2$, $C=c_0+\theta^tC_1$, $c_0\neq 0$, $c_{10}\neq 0$.

4 $s+2<\mu_2$, then since $c_0\neq 0$, $1-C(\theta C)=-\theta^{\mu_2-s-2+r}C_2$, or $C(\theta C)=1+\theta^{\mu_2-s-2+r}C_2$, $c_{20}\neq 0$. $\therefore$ $c_0+\theta^tC_1(\theta C)C^t=1+\theta^{\mu_2-s-2+r}C_2$,

or $$c_0+\theta^t[c_{10}+\theta C\cdot C_3(\theta C)][c_0^t+\theta^tC_4]=1+\theta^{\mu_2-s-2+r}C_2,$$

or $$c_0+\theta^tc_{10}c_0^t+\theta^{t+1}C_5=1+\theta^{\mu_2-s-2+r}C_2.$$

Therefore $c_0=1$ and $t=\mu_2-s-2+r$ since $c_0^t\neq 0\neq c_{10}$ and $C(\theta C)=1+\theta^t C_2$, $t<\mu_2$. Consider next equation (3)′,

$$\delta^{\mu_1-\mu_2+s+1}B[1-C^{s+1}]=0.$$

For class 3, $s+2=\mu_2$, $C=c_0+\theta^t C_1$, $c_0\neq 0$, $c_{10}\neq 0$ this becomes

$$\delta^{\mu_1-1}b_0[1-C^{\mu_2-1}]=0,$$

from which we have either

31 $\qquad b_0=0, \quad B=\delta B_1, \quad c_0^{\mu_2-1}\neq 1,$

or 32 $\qquad c_0^{\mu_2-1}=1.$

For class 4, $s+2<\mu_2$, $c_0=1$, $t=\mu_2-s-2+r$ equation (3)′ becomes

$$\delta^{\mu_1-\mu_2+s+1}B[1-(1+\delta^{\mu_2-s-2+r}C_1)^{s+1}]=0,$$
$$\delta^{\mu_1-1+r}B[-C_1(s+1)+\ldots.]=0, \quad \delta^{\mu_1-1+r}B=0.$$

This is satisfied unless $r=0$, then $B=\delta B_1$. Consider now equation (2)′,

$$\delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0.$$

For 31 $s+2=\mu_2$, $c_0^{\mu_2-1}\neq 1$, $C=c_0+\delta^t C_1$, $c_0\neq 0\neq c_{10}$. Put $B=\delta^v B_2$, $b_{20}\neq 0$. Then $\delta^{\mu_1-\mu_2+v+1}B_2[1-C\cdot C(\delta C)]=0=\delta^{\mu_1-\mu_2+v+1}[1-C\cdot C(\delta C)]$.

311 If $c_0^2\neq 1$ then $v\geq\mu_2-1$, $B=\delta^{\mu_2-1}b_{\mu_2-1}$, or $B=0$ if $b_{\mu_2-1}=0$.

312 If $c_0^2=1$, $c_0^{\mu_2-1}\neq 1$, that is $c_0=-1$ and μ_2-1 is odd, then

$$\mu_1-\mu_2+v+t+1\geq\mu_1. \quad \therefore\ v\geq\mu_2-t-1.$$

For 32 $\qquad c_0^{\mu_2-1}=1$, then $\delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0$,

$$\delta^{\mu_1-\mu_2+1}B[1-(c_0+\delta^t C_1)(c_0+\delta^t C^t C_1(\delta C)]=0.$$

321 If $c_0^2\neq 1$, $B=\delta^{\mu_2-1}b_{\mu_2-1}$.

322 If $c_0^2=1$, $B=\delta^{\mu_2-t-1}B_3$, and if $c_0=-1$, μ_2-1 is even, that is μ_2 is odd.

For Class 4, $s+2<\mu_2$, $c_0=1$, $t=\mu_2-s-2+r$, equation (2)′ becomes

$$\delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0, \quad \delta^{\mu_1-\mu_2+1}B[1-(1+\delta^t C_1)(1+\delta^t C^t C_1(\delta C)]=0,$$
$$\therefore\ \delta^{\mu_1-\mu_2+1+t}B=0, \text{ which is } \delta^{\mu_1-s-1+r}B=0. \quad \therefore\ B=\delta^{s+1-r}B_4.$$

We consider finally equation (4)′,

$$\theta^{2s+2}[1-C^{s+1}]+\theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0.$$

For 311 $c_0^2\neq 1$, $B=\delta^{\mu_2-1}b$ where b may be 0, $s+2=\mu_2$, $c_0^{\mu_2-1}\neq 1$, $c_0\neq 0\neq c_{10}$, then $\theta^{\mu_1-1}b[1-(c_0+\theta^t C_1)^{\mu_1-1}]=0$, which is satisfied unless $\mu_1=\mu_2$ since $\theta^{\mu_2}=0$. Therefore we have also

3111 $\mu_1=\mu_2$, $b=b^*_{\mu_2-1}=0$, *i. e.*, $v>\mu_2-1$, and

3112 $\mu_1=\mu_2$, $c_0^{\mu_1-1}=1$, but this is impossible since it is a contradiction to 311.

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* We write b in place of b_{μ_2-1} in what follows.

For 312 $c_0=-1$, μ_2-1 is odd, $B=\theta^v B_2$, $b_{20}\neq 0$, $v>\mu_2-t-1$, equation (4)′ becomes

$$\theta^{\mu_1-\mu_2+v}[B_2-(-1+\theta C_1)^{\mu_1-\mu_2+v}B_2(\theta C)]=0.$$

We have then either

3121, $\mu_1-\mu_2+v$ is odd, and $\mu_1-\mu_2+v\geq\mu_2$, *i. e.*, $v\geq 2\mu_2-\mu_1$, or
3122, $\mu_1-\mu_2+v$ is even.

Now $\qquad \theta^{\mu_1-\mu_2+v}[B_2-B_2(\theta C)-\ldots\ldots]=0$

becomes $\qquad \theta^{\mu_1-\mu_2+v+1}[b_{21}(1-C)+\ldots\ldots]=0,$

and $\qquad \mu_1-\mu_2+v+1\geq\mu_2$, *i. e.*, $v\geq 2\mu_2-\mu_1-1.$

For 321 $c_0^2\neq 1$, $B=\delta^{\mu_2-1}b_{\mu_2-1}$, $c_0^{\mu_2-1}=1$, equation (4)′ becomes

$$\theta^{\mu_1-1}b_{\mu_2-1}[1-(c_0+\theta^t C_1)^{\mu_1-1}]=0$$

which is satisfied since $\theta^{\mu_2}=0$. If $\mu_1>\mu_2$, $\theta^{\mu_1-1}=0$. If $\mu_1=\mu_2$, the [] causes the expression to vanish.

For 322 $c_0^2=1$, $B=\delta^{\mu_2-t-1}B_3$, μ_2-1 is even if $c_0=-1$ and (4)′ becomes $\theta^{\mu_1-t-1}[B_3-(c_0+\theta^t C_1)^{\mu_1-t-1}B_3(\theta C)]=0$, for which we have

3221, $c_0=1$, $\theta^{\mu_1-t-1}[b_{31}\theta(1-C)+\ldots\ldots-\theta^t C_1 B_3(\theta C)-\ldots\ldots]=0,$
$\therefore$ $\theta^{\mu_1-1}B_3(\theta C)=0$, which is satisfied unless $\mu_1=\mu_2$. Then

32211, $\mu_1>\mu_2$.

32212, $\mu_1=\mu_2$, $B_3(\theta C)=\theta B_5$, *i. e.*, $B=\theta^{\mu_2-t}B$.

3222, $c_0=-1$, μ_2-1 is even, then

$$\theta^{\mu_1-t-1}[B_3-(-1)^{\mu_1-t-1}B_3(\theta C)-(-1)^{\mu_1-t}\theta^t C_1 B_3(\theta C)+\ldots\ldots]=0.$$

32221. If μ_1-t-1 is even, $\theta^{\mu_1-t-1}[b_{31}\theta(1-C)+\ldots\ldots]=0.$
$\therefore$ $B_3=\theta^t B_6$, which makes $B=\theta^{\mu_2-1}b_{\mu_2-1}$.

32222. If μ_1-t-1 is odd, $B_3=\theta^{t+1}B_7$, which makes $B=0$.

For Class 4 $s+2<\mu_2$, $c_0=1$, $t=\mu_2-s-2+r$, $B=\theta^{s+1-r}B_4$.
Equation (4)′ becomes

$$\theta^{2s+2}[1-C^{s+1}]+\theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0,$$
$$\theta^{\mu_1-\mu_2+s+1-r}[B_4-(1+\theta^t C_1)^{\mu_1-\mu_2+s+1-r}B_4(\theta C)]=0,$$
$$\theta^{\mu_1-\mu_2+s+1-r}[b_{41}\theta(1-C)+\ldots\ldots+\theta^t C_1 B_4(\theta C)+\ldots\ldots]=0,$$
$$\theta^{\mu_1-1}[b_{41}\theta+b_{42}\theta^2(1+C)+\ldots\ldots+C_1 B_4(\theta C)+\ldots\ldots]=0,$$

$\therefore$ $\theta^{\mu_1-1}B_4(\theta C)=0$, for which we have either

41, $\mu_1>\mu_2$, or

42, $\mu_1=\mu_2$, $B_4=\theta B_6$, *i. e.*, $B=\theta^{s+2-r}B_6$.

We have therefore in Case III_1 the following eleven types:

$$s+2=\mu_2 \begin{cases} 311. & \mu_1>\mu_2, \quad B=\theta^{\mu_2-1}b_{\mu_2-1}, \quad c_0^{\mu_2-1}\neq 1, \quad c_0^2\neq 1. \\ 3111. & \mu_1=\mu_2, \quad B=0, \quad c_0^2\neq 1, \quad c_0^{\mu_2-1}\neq 1. \\ 3121. & c_0=-1, \ c_0^{\mu_2-1}\neq 1, \ \mu_2-1 \text{ is odd}, \ B=\theta^v B_2, \ \mu_1-\mu_2+v \text{ is odd}, \\ & v\geq 2\mu_2-\mu_1, \quad v>\mu_2-t-1, \quad \mu_1-\mu_2\geq t+1. \\ 3122. & c_0=-1, \ c_0^{\mu_2-1}\neq 1, \ \mu_2-1 \text{ is odd}, \ B=\theta^v B_2, \ \mu_1-\mu_2+v \text{ is even}, \\ & v\geq 2\mu_2-\mu_1-1. \\ 321. & c_0^2\neq 1, \quad c_0^{\mu_2-1}=1, \quad B=\theta^{\mu_2-1}b_{\mu_2-1}. \\ 32211. & \mu_1>\mu_2, \quad c_0=1, \quad B=\theta^{\mu_2-t-1}B_3. \\ 32212. & \mu_1=\mu_2, \quad c_0=1, \quad B=\theta^{\mu_2-t}B_5. \\ 32221. & c_0=-1, \ \mu_2-1 \text{ is even}, \ \mu_1-t-1 \text{ is even}, \ B=\theta^{\mu_2-1}b_{\mu_2-1}. \\ 32222. & c_0=-1, \ \mu_2-1 \text{ is even}, \ \mu_1-t-1 \text{ is odd}, \ B=0. \end{cases}$$

$$s+2>\mu_2 \begin{cases} 41. & \mu_1>\mu_2, \quad c_0=1, \quad t=\mu_2-s-2+r, \quad B=\theta^{s+1-r}B_4. \\ 42. & \mu_1=\mu_2, \quad c_0=1, \quad t=\mu_2-s-2+r, \quad B=\theta^{s+2-r}B_6. \end{cases}$$

These particular types are written out as follows:

311. $s+2=\mu_2, \quad \mu_1>\mu_2, \quad c_0^2\neq 1, \quad B=\theta^{\mu_2-1}b_{\mu_2-1}, \quad C=c_0+\theta^t C_1.$

$$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-1}b, \quad j=111+22\theta C.$$

Then $ij=211, \quad ji=21\theta C=ijC(j)=c_0ij+ij^{t+1}C_1(j),$

$$i^2=21\theta^{\mu_2-1}+11\delta^{\mu_1-1}b=ij^{\mu_2-1}+j^{\mu_1-1}b.$$

3111. $s+2=\mu_2, \quad \mu_1=\mu_2, \quad B=0, \quad c_0^2\neq 1, \quad c^{\mu_2-1}\neq 1.$

$$i=210+22\theta^{\mu_2-1}, \quad j=111+22\theta C.$$

Then $ij=211, \quad ji=21\theta C=ijC(j)=c_0ij+ij^{t+1}C_1(j), \quad i^2=21\theta^{\mu_2-1}=ij^{\mu_2-1}.$

3121. $c_0=-1, \ c_0^{\mu_2-1}\neq 1, \ \mu_2-1$ is odd, $B=\theta^v B_2, \ \mu_1-\mu_2+v$ is odd, $v\geq 2\mu_2-\mu_1,$ $v>\mu_2-t-1, \ \mu_1-\mu_2\geq t+1.$

$$\begin{aligned} i&=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-\mu_2+v}B_2, \\ j&=111+22(-1)+22\theta^{t+1}C_1=111+22\theta C, \quad ij^n=21n+12\delta^{\mu_1-\mu_2+v+n}(-1)^n B_2, \\ ji&=211(-1)+12\delta^{\mu_1-\mu_2+v+1}B_2+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j), \\ j^n&=11n+22\theta^n C^n=11n+22\theta^n(-1+\theta^t C_1)^n, \quad j^{\mu_1-\mu_2+v}=11\delta^{\mu_1-\mu_2+v}, \\ i^2&=21\theta^{\mu_2-1}+22\theta^{\mu_1-\mu_2+v}B_2+12\delta^{\mu_1+v-1}B_2+11\delta^{\mu_1-\mu_2+v}B_2 \\ &=ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j). \end{aligned}$$

3122. $c_0=-1, \quad c_0^{\mu_2-1}\neq 1, \quad \mu_2-1$ is odd, $B=\theta^v B_2, \quad \mu_1-\mu_2+v$ is even, $v\geq 2\mu_2-\mu_1-1, \ v>\mu_2-t-1.$

$$\begin{aligned} i&=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-\mu_2+v}B_2, \\ j&=111+221(-1)+22\theta^{t+1}C_1=111+22\theta C, \quad ij^n=21n+12\delta^{\mu_1-\mu_2+v+n}(-1)^n B_2, \\ ji&=211(-1)+21\theta^{t+1}C_1+12\delta^{\mu_1-\mu_2+v+1}B_2=-ij+ij^{t+1}C_1(j), \\ j^n&=11n+22\theta^n(-1+\theta^t C_1)^n, \quad j^{\mu_1-\mu_2+v}=11\delta^{\mu_1-\mu_2+v}, \\ i^2&=21\theta^{\mu_2-1}+11\delta^{\mu_1-\mu_2+v}B_2+12\delta^{\mu_1+v-1}B_2=ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j). \end{aligned}$$

321. $c_0^2 \neq 1, \quad c_0^{\mu_2-1}=1, \quad B=\theta^{\mu_2-1}b, \quad s+2=\mu_2.$

$$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-1}b, \quad j=111+22\theta C, \quad ij=211,$$
$$ji=21\theta C=ijC(j); \quad i^2=21\theta^{\mu_2-1}+22\theta^{\mu_1-1}b+11\delta^{\mu_1-1}b=ij^{\mu_2-1}+j^{\mu_1-1}b.$$

32211. $c_0=1, \quad \mu_1>\mu_2, \quad B=\theta^{\mu_2-t-1}B_8, \quad s+2=\mu_2.$

$$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-t-1}B_8,$$
$$j=111+221+22\theta^{t+1}C_1, \quad ij^n=21n+12\delta^{\mu_1-t+n-1}B_8,$$
$$ji=211+12\delta^{\mu_1-t}B_8+21\theta^{t+1}C_1=ij+ij^{t+1}C_1(j), \quad j^n=11n+22\theta^n(1+\theta^t C_1)^n,$$
$$i^2=21\theta^{\mu_2-1}+22\theta^{\mu_1-t-1}B_8+11\delta^{\mu_1-t-1}B_8+12\delta^{\mu_1-\mu_2-t-2}B_8$$
$$=ij^{\mu_2-1}-12\delta^{\mu_1-\mu_2-t-2}B_8+j^{\mu_1-t-1}B_8-22\theta^{\mu_1-t-1}B_8$$
$$+12\delta^{\mu_1-\mu_2-t-2}B_8+22\theta^{\mu_1-t-1}B_8=ij^{\mu_2-1}+j^{\mu_1-t-1}B_8(j).$$

32212. $c_0=1, \quad \mu_1=\mu_2, \quad B=\theta^{\mu_2-t}B_5, \quad s+2=\mu_2.$

$$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-t}B_5, \quad j=111+221+22\theta^{t+1}C_1,$$
$$ij^n=21n+12\delta^{\mu_1-t+n}B_5, \quad ji=211+21\theta^{t+1}C_1+12\delta^{\mu_1-t+1}B_5=ij+ij^{t+1}C_1(j),$$
$$j^n=11n+22\theta^n(1+\theta^t C_1)^n, \quad i^2=22\theta^{\mu_1-t}B_5+21\theta^{\mu_2-1}+11\delta^{\mu_1-t}B_5,$$
$$i^2=ij^{\mu_2-1}+j^{\mu_1-t}B_5(j).$$

32221. $c_0=-1$, μ_2-1 is even, μ_1-t-1 is even, $B=\theta^{\mu_2-1}b$, $s+2=\mu_2.$

$$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-1}b, \quad j=111+221(-1)+22\theta^{t+1}C_1, \quad ij=211,$$
$$ji=211(-1)+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j), \quad j^n=11n+22\theta^n(-1+\theta^t C_1)^n,$$
$$i^2=21\theta^{\mu_2-1}+22\theta^{\mu_1-1}b+11\delta^{\mu_1-1}b$$
$$=ij^{\mu_2-1}+j^{\mu_1-1}b-22\theta^{\mu_1-1}b(-1)^{\mu_1-1}+22\theta^{\mu_1-1}b.$$

Since $\theta^{\mu_1-1}=0$, unless $\mu_1=\mu_2$, these last two terms vanish separately. If $\mu_1=\mu_2$ they destroy each other since $(-1)^{\mu_2-1}=1$.

32222. $c_0=-1$, μ_2-1 is even, μ_1-t-1 is odd, $B=0$, $s+2=\mu_2.$

$$i=210+22\theta^{\mu_2-1}, \quad j=111+221(-1)+22\theta^{t+1}C_1, \quad ij=211,$$
$$ji=211(-1)+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j), \quad i^2=21\theta^{\mu_2-1}=ij^{\mu_2-1}.$$

41. $s+2<\mu_2, \quad \mu_1>\mu_2, \quad c_0=1, \quad t=\mu_2-s-2+r, \quad B=\theta^{s+1-r}B_4.$

$$i=210+22\theta^{s+1}+12\delta^{\mu_1-\mu_2+s+1-r}B_4, \quad j=111+221+22\theta^{t+1}C_1=111+22\theta C.$$
$$ij^n=21n+12\delta^{\mu_1-t+n-1}C^nB_4+22\theta^{s+n+1}C^n,$$
$$ji=21\theta C+22\theta^{s+2}C+12\delta^{\mu_1-t}B_4=ijC(j)=ij+ij^{t+1}C_1(j).$$
$$i^2=21\theta^{s+1}+22\theta^{\mu_1-t-1}B_4+11\delta^{\mu_1-t-1}B_4+12\delta^{\mu_1-t+s}B_4+22\theta^{2s+2}$$
$$=ij^{s+1}+j^{\mu_1-t-1}B_4.$$

42. $s+2<\mu_2, \quad \mu_1=\mu_2, \quad c_0=1, \quad t=\mu_2-s-2+r, \quad B=\theta^{s+2-r}B_6.$

$$i=210+22\theta^{s+1}+12\delta^{s+2-r}B_6, \quad j=111+22\theta C,$$
$$ij^n=21n+22\theta^{s+n+1}C^n+12\delta^{s+n+2-r}B_6C^n,$$
$$ji=21\theta C+22\theta^{s+2}C+12\delta^{s+2-r}B_6=ijC(j)=ij+ij^{t+1}C_1(j),$$
$$ij^2=21\theta^{s+1}+22\theta^{2s+2}+22\theta^{s+2-r}B_6+11\delta^{s+2-r}B_6+12\delta^{2s+3-r}B_6$$
$$=ij^{s+1}+j^{s+2-r}B_6(j).$$

Type	$ji=$	$i^2=$	
1	$ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-t-1}B_3(j)$,	$\mu_1>\mu_2$,
2	$ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-t}B_5(j)$,	$\mu_1=\mu_2$,
3	$ij+ij^{t+1}C_1(j)$,	$ij^{s+1}+j^{\mu_1-t-1}B_4(j)$,	$\mu_1>\mu_2$,
4	$ij+ij^{t+1}C_1(j)$,	$ij^{s+1}+j^{\mu_2-t}B_6(j)$,	$\mu_1=\mu_2$,
5	$c_0ij+ij^{t+1}C_1(j)$,	ij^{μ_2-1},	$c_0^2\neq 1$, $\mu_1=\mu_2$,
6	$c_0ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-1}b$,	$c_0^2\neq 1$, $\mu_1>\mu_2$,
7	$c_0ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-1}b$,	$c_0^2\neq 1$, $c_0^{\mu_2-1}=1$,
8	$-ij+ij^{t+1}C_1(j)$,	ij^{μ_2-1},	μ_1-t is even, μ_2 is odd,
9	$-ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-1}b$,	μ_1-t is odd, μ_2 is odd,
10	$-ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j)$,	$v\geq 2\mu_2-\mu_1$, μ_1-v is odd,
11	$-ij+ij^{t+1}C_1(j)$,	$ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j)$,	$v\geq 2\mu_2-\mu_1-1$, μ_1-v is even.

Case III_2. $c_0=0$.

$C=\theta^t C_1(\theta)$, $0<t\leq\mu_2-2$. We go back to equations (A) which now become,

(1) $\theta^{t+2}A[1-C(\theta C)]=0$, for which we have either

1. $t=\mu_2-2$, $C=\theta^{\mu_2-2}c$, where c is written for c_{μ_2-2}, or
2. $A=\theta^{\mu_2-t-2}A_1$.

(2) $\delta^{\mu_1-\mu_2+1}B=0$. $\therefore$ $B=\delta^{\mu_2-1}b$, where b may be zero.

(3) $\delta^{\mu_1-\mu_2+1}B[A-C\cdot A(\theta C)]=0$. This is satisfied by the result of (2).

We now use the results of equations (1) and (2) in (4). For 1 $t=\mu_2-2$, $C=\theta^{\mu_2-2}c$, $B=\delta^{\mu_2-1}b$, equation (4) becomes

(4) $\theta^2 A[A-\theta^{\mu_2-2}cA(\theta C)]+\theta^{\mu_1-1}b[1-C^{\mu_1-1}]=0$, $\theta^2A^2+\theta^{\mu_1-1}b=0$.

The three possible solutions of this are

11, $\mu_1>\mu_2$, $A=\theta^{m-1}A_2$ where $m=\mu_2/2$ if μ_2 is even,

$$m=\frac{\mu_2-1}{2} \text{ if } \mu_2 \text{ is odd,}$$

12, $\mu_1=\mu_2=\mu$, $A=\theta^{\frac{\mu-2}{2}}A_3$, $B=0$, μ is even.

13, $\mu_1=\mu_2=\mu$, $A=\theta^{\frac{\mu-3}{2}}A_4$, $B=\delta^{\mu_2-1}b$, $a_{40}^2+b=0$, μ is odd.

For 2 $C=\theta^tC_1$, $B=\delta^{\mu_2-1}b$, $A=\theta^{\mu_2-t-2}A_1$, (4) becomes

$$(4)\qquad \theta^{2\mu_2-2t-2}A_1[A_1-C^{\mu_2-t-1}A_1(\theta C)]+\theta^{\mu_1-1}b=0,$$
$$\theta^{2(\mu_2-t-1)}A_1^2-\theta^{(\mu_2-t-1)t+2\mu_2-2t-2}A_1C_1A_1(\theta C)+\theta^{\mu_1-1}b=0.$$

This reduces to

$$\theta^{2(\mu_2-t-1)}A_1^2+\theta^{\mu_1-1}b=0,$$

because $(\mu_2-t-1)t+2\mu_2-2t-2\geq\mu_2$, i. e., $(t+2)\mu_2-(t^2+3t+2)\geq\mu_2$, or $\mu_2-t-2\geq 0$, $t\leq\mu_2-2$, which is true for this case.

Now the possible solutions of

$$\theta^{2(\mu_2-t-1)}A_1^2+\theta^{\mu_1-1}b=0$$

are 21, $\mu_1>\mu_2$, which gives rise to

211, $2t\leq\mu_2-2$,

and 212, $2t>\mu_2-2$, $A_1=\theta^{t+1-m}A_2$, $m=\frac{\mu_2}{2}$ if μ_2 is even,

$m=\frac{\mu_2+1}{2}$ if μ_2 is odd,

and 22, $\mu_1=\mu_2=\mu$, which gives rise to the following:

221, $2t\leq\mu_2-2$, $b=0$,

222, $2t>\mu_2-2$, $A_1=\theta^{t+1-\frac{\mu}{2}}A_2$, $b=0$, μ is even,

223, $2t>\mu-2$, $A_1=\theta^{t+1-\frac{\mu-1}{2}}A_2'$, $a_{20}^2+b=0$, μ is odd.

Therefore we have the following eight types:

11. $\mu_1>\mu_2$, $A=\theta^{m-1}A_2$, $B=\theta^{\mu_1-1}b$, $C=\theta^{\mu_2-2}c$, $m=\frac{\mu_2}{2}$ if μ_2 is even.

$m=\frac{\mu_2-1}{2}$ if μ_2 is odd.

12. $\mu_1=\mu_2=\mu$ is even, $A=\theta^{\frac{\mu-2}{2}}A_3$, $B=0$, $C=\theta^{\mu-2}c$.

13. $\mu_1=\mu_2=\mu$ is odd, $A=\theta^{\frac{\mu-3}{2}}A_4$, $B=\theta^{\mu-1}b$, $C=\theta^{\mu-2}c$, $a_{40}^2+b=0$.

211. $\mu_1>\mu_2$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-t-2}A_1$, $B=\delta^{\mu_1-1}b$, $C=\theta^tC_1$.

212. $\mu_1>\mu_2$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-m-1}A_2$, $B=\delta^{\mu_1-1}b$, $C=\theta^tC_1$,

$m=\frac{\mu_2}{2}$ if μ_2 is even, $m=\frac{\mu_2+1}{2}$ if μ_2 is odd.

221. $\mu_1=\mu_2=\mu$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-t-2}A_1$, $B=0$, $C=\theta^tC_1$.

222. $\mu_1=\mu_2=\mu$ is even, $2t>\mu-2$, $A=\theta^{\frac{\mu}{2}-1}A_2$, $B=0$, $C=\theta^tC_1$.

223. $\mu_1=\mu_2=\mu$ is odd, $2t>\mu-2$, $A=\theta^{\frac{\mu-1}{2}-1}A_2'$, $B=\delta^{\mu-1}b$, $C=\theta^tC_1$, $a_{20}^2+b=0$.

These eight written out give the following:

11. $\mu_1>\mu_2$, $A=\theta^{m-1}A_2$, $B=\delta^{\mu_1-1}b$, $C=\theta^{\mu_2-2}C_1$, $m=\frac{\mu_2}{2}$ if μ_2 is even,

$m=\frac{\mu_2-1}{2}$ if μ_2 is odd.

$i=210+22\theta^mA_2+12\delta^{\mu_1-1}b$, $j=111+22\theta^{\mu_2-1}c$.

Hence $ij=211$, $ji=21\theta^{\mu_2-1}c=ij^{\mu_2-1}c$, $i^2=21\theta^mA_2+11\delta^{\mu_1-1}b=ij^mA_2(j)+j^{\mu_1-1}b$.

12. $\mu_1=\mu_2=\mu$ is even, $A=\theta^{\frac{\mu-2}{2}}A_3$, $B=0$, $C=\theta^{\mu-2}c$.

$i=210+22\theta^{\frac{\mu}{2}}A_3$, $j=111+22\theta^{\mu-1}c$, $ij=211$,

$ji=21\theta^{\mu-1}c=ij^{\mu-1}c$, $i^2=21\theta^{\frac{\mu}{2}}A_3=ij^{\frac{\mu}{2}}A_3(j)$.

13. $\mu_1=\mu_2=\mu$ is odd, $A=\theta^{\frac{\mu-3}{2}}A_4$, $B=\delta^{\mu_1-1}b$, $C=\theta^{\mu-2}c$, $a_{40}^2+b=0$.

$i=210+22\theta^{\frac{\mu-1}{2}}A_4+12\delta^{\mu-1}b$, $j=111+22\theta^{\mu-1}c$, $ij=211$,

$ji=21\theta^{\mu-1}c=ij^{\mu-1}c$,

$i^2=21\theta^{\frac{\mu-1}{2}}A_4+22\theta^{\mu-1}a_{40}^2+22\theta^{\mu-1}b+11\delta^{\mu-1}b=ij^{\frac{\mu-1}{2}}A_4(j)+j^{\mu-1}b$.

211. $\mu_1>\mu_2$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-t-2}A_1$, $B=\delta^{\mu_1-1}b$, $C=\theta^tC_1$.

$i=210+22\theta^{\mu_2-t-1}A_1+12\delta^{\mu_1-1}b$, $j=111+22\theta^{t+1}C_1$, $ij=211$,

$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$, $i^2=21\theta^{\mu_2-t-1}A_1+11\delta^{\mu_1-1}b=ij^{\mu_2-t-1}A_1(j)+j^{\mu_1-1}b$.

212. $\mu_1>\mu_2$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-m-1}A_2$, $B=\delta^{\mu_1-1}b$, $C=\theta^tC_1$,

$m=\frac{\mu_2}{2}$ if μ_2 is even, $m=\frac{\mu_2+1}{2}$ if μ_2 is odd.

$i=210+22\theta^{\mu_2-m}A_2+12\delta^{\mu_1-1}b$, $j=111+22\theta^{t+1}C_1$, $ij=211$,

$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$, $i^2=21\theta^{\mu_2-m}A_2+11\delta^{\mu_1-1}b=ij^{\mu_2-m}A_2(j)+j^{\mu_1-1}b$.

221. $\mu_1=\mu_2=\mu$, $2t\leq\mu_1-2$, $A=\theta^{\mu-t-2}A_1$, $B=0$, $C=\theta^tC_1$.

$i=210+22\theta^{\mu-t-1}A_1$, $j=111+22\theta^{t+1}C_1$, $ij=211$,

$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$, $i^2=21\theta^{\mu-t-1}A_1=ij^{\mu-t-1}A_1(j)$.

222. μ is even, $2t>\mu-2$, $A=\theta^{\frac{\mu}{2}-1}A_2$, $B=0$, $C=\theta^tC_1$.

$i=210+22\theta^{\frac{\mu}{2}}A_2$, $j=111+22\theta^{t+1}C_1$, $ij=211$,

$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$, $i^2=21\theta^{\frac{\mu}{2}}A_2=ij^{\frac{\mu}{2}}A_2(j)$.

223. μ is odd, $2t>\mu-2$, $A=\theta^{\frac{\mu-1}{2}-1}A_2'$, $B=\delta^{\mu-1}b$, $C=\theta^tC_1$, $a_{20}'^2+b=0$.

$i=210+22\theta^{\frac{\mu-1}{2}}A_2'+12\delta^{\mu-1}b$, $j=111+22\theta^{t+1}C_1$, $ij=211$,

$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$,

$i^2=21\theta^{\frac{\mu-1}{2}}A_2'+11\delta^{\mu-1}b+22\theta^{\mu-1}b+22\theta^{\mu-1}a_{20}'^2=ij^{\frac{\mu-1}{2}}A_2'(j)+j^{\mu-1}b$.

Type	$ji=$	$i^2=$	
1	$ij^{\mu_2-1}c$,	$ij^mA_2(j)+j^{\mu_1-1}b$,	$\mu_1>\mu_2$,
2	$ij^{\mu-1}c$,	$ij^{\frac{\mu}{2}}A_3(j)$,	$\mu_1=\mu_2=\mu$ is even,
3	$ij^{\mu-1}c$,	$ij^{\frac{\mu-1}{2}}A_4(j)+bj^{\mu-1}$,	μ is odd, $b+a_{40}^2=0$,
4	$ij^{t+1}C_1(j)$,	$ij^{\mu_2-t-1}A_1(j)+bj^{\mu_1-1}$,	
5	$ij^{t+1}C_1(j)$,	$ij^{\mu_2-m}A_2(j)+bj^{\mu_1-1}$,	$m=\frac{\mu_2}{2}$ if μ_2 is even, $m=\frac{\mu_2+1}{2}$ if μ_2 is odd.
6	$ij^{t+1}C_1(j)$,	$ij^{\mu-t-1}A_1(j)$,	$2t\leq\mu-2$,
7	$ij^{t+1}C_1(j)$,	$ij^{\frac{\mu}{2}}A_2(j)$,	$2t>\mu-2$, μ is even,
8	$ij^{t+1}C_1(j)$,	$ij^{\frac{\mu-1}{2}}A_2'(j)+j^{\mu-1}b$,	$b+a_{20}'^2=0$, $2t>\mu-2$, μ is odd.

VII. *Complete List of the Types Arranged According to the Form of the Product ji, with Sufficient Conditions to Distinguish Each.*

Class	Type	$ji=$	$i^2=$	Conditions
I	1	$ij,$	$i'^2=j^{\mu_1-\mu_2}B(j),$	$E=0.$
	2	$ij+j^{\mu_2+1}E_1(j),$	$i'^2=j^{\mu_2}B(j),$	$\mu_1-\mu_2=\mu_2,\ E_1\neq 0,$
II	3	$ij+j^{\mu_2+\alpha}E(j),$	$i'^2=j^{\mu_2+\alpha}B(j),$	$\mu_1-\mu_2=\mu_2+\alpha,\ \alpha>0,\ E\neq 0,$
	4	$ij+j^{\mu_1-\mu_2+v}E_3(j),$	$i'^2=j^{\mu_1-\mu_2}B(j),$	$\mu_1-\mu_2<\mu_2,\ v>2\mu_2-\mu_1,\ E_3\neq 0.$
	5	$ij+ij^{t+1}C_1(j),$	$j^{\mu_1-t}B_2(j),$	$\mu_1=\mu_2,$
	6	$ij+ij^{t+1}C_1(j),$	$j^{\mu_1-t-1}B_1(j),$	$\mu_1>\mu_2,$
III	7	$ij+ij^{t+1}C_1(j),$	$ij^{\mu_2-1}+j^{\mu_1-t-1}B_3(j),$	$\mu_1>\mu_2,$
	8	$ij+ij^{t+1}C_1(j),$	$ij^{\mu_2-1}+j^{\mu_1-t}B_5(j),$	$\mu_1=\mu_2,$
	9	$ij+ij^{t+1}C_1(j),$	$ij^{s+1}+j^{\mu_1-t-1}B_4(j),$	$\mu_1>\mu_2,$
	10	$ij+ij^{t+1}C_1(j),$	$ij^{s+1}+j^{\mu_2-t}B_6(j),$	$\mu_1=\mu_2.$
	11	$c_0ij+ij^{t+1}C_1(j),$	$0,$	$c_0^2\neq 1,$
	12	$c_0ij+ij^{t+1}C_1(j),$	$j^{\mu_1-1}b,$	$c_0^2\neq 1,\ b\neq 0,$
IV	13	$c_0ij+ij^{t+1}C_1(j),$	$ij^{\mu_2-1}+j^{\mu_1-1}b,$	$c_0^2\neq 1,\ \mu_1>\mu_2,$
	14	$c_0ij+ij^{t+1}C_1(j),$	$ij^{\mu_2-1},$	$c_0^2\neq 1,\ \mu_1=\mu_2,$
	15	$c_0ij+ij^{t+1}C_1(j),$	$ij^{\mu_2-1}+j^{\mu_1-1}b,$	$c_0^2\neq 1,\ c_0^{\mu_2-1}=1.$
	16	$-ij+ij^{t+1}C_1(j),$	$0,$	
	17	$-ij+ij^{t+1}C_1(j),$	$j^{\mu_1-1}b,$	$b\neq 0,$
	18	$-ij+ij^{t+1}C_1(j),$	$ij^{\mu_2-1},$	μ_1-t is even, μ_2 is odd,
V	19	$-ij+ij^{t+1}C_1(j),$	$j^{\mu_1-t-1}B_1(j),$	$B_1(j)\neq 0,$
	20	$-ij+ij^{t+1}C_1(j),$	$ij^{\mu_2-1}+j^{\mu_1-1}b,$	μ_1-t is odd, μ_2 is odd,
	21	$-ij+ij^{t+1}C_1(j),$	$ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j),$	$v\geq 2\mu_2-\mu_1,\ \mu_1-v$ is odd,
	22	$-ij+ij^{t+1}C_1(j),$	$ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j),$	$v\geq 2\mu_2-\mu_1-1,\ \mu_1-v$ is even.
	23	$ij^{\mu_2-1}c,$	$ij^mA_2(j)+j^{\mu_1-1}b;$	$\mu_1>\mu_2$
	24	$ij^{\mu-1}c,$	$ij^{\frac{\mu}{2}}A_3(j),$	$\mu_1=\mu_2=\mu$ is even,
	25	$ij^{\mu-1}c,$	$ij^{\frac{\mu-1}{2}}A_4(j)+bj^{\mu-1},$	$b+a_{40}^2=0,\ \mu$ is odd,
	26	$ij^{t+1}C_1(j),$	$ij^{\mu_2-t-1}A_1(j)+bj^{\mu_1-1},$	
VI	27	$ij^{t+1}C_1(j),$	$ij^{\mu_2-m}A_2(j)+bj^{\mu_1-1},$	$m=\dfrac{\mu_2}{2}$ if μ_2 is even,
				$m=\dfrac{\mu_2+1}{2}$ if μ_2 is odd,
	28	$ij^{t+1}C_1(j),$	$ij^{\mu-t-1}A_1(j),$	$2t\leq\mu-2,$
	29	$ij^{t+1}C_1(j),$	$ij^{\frac{\mu}{2}}A_2(j),$	$2t>\mu-2,\ \mu$ is even,
	30	$ij^{t+1}C_1(j),$	$ij^{\frac{\mu-1}{2}}A_2'(j)+bj^{\mu-1},$	$2t>\mu-2,\ \mu$ is odd, $b+a_{20}'^2=0.$

These types when arranged according to the form of the product ji fall into six classes.

Class I, made up of type 1, is the only commutative type of the entire set.

Class II, types 2–4, have for the product ji the term ij and powers of j.

Class III, types 5–10, have $ji = ij + ij^{t+1}C_1(j)$ where $t > 0$ and $C_1 \neq 0$.

Class IV, types 11–15, differ from those of Class III in that

$$ji = c_0 ij + ij^{t+1}C_1(j) \quad \text{where } c_0^2 \neq 1.$$

Class V, types 16–22, are essentially different from those of Classes III and IV since $ji = -ij + ij^{t+1}C_1(j)$.

Class VI, types 23–30, have the product ji starting with at least ij^2.

We need now to see that the types of each class are essentially distinct. The conditions given, having arisen in the determination of these types, are sufficient to show this. For example, we might think that type 18 and type 20 would be the same if in type 20 $b = 0$. Although this would make the products ji and i^2 have the same expressions for the two types, they would still be distinct, for in 18 $\mu_1 - t$ is odd and in 20 it is even. Types 13 and 15 have the same form for ji and i^2, but they are essentially distinct because in 13 $\mu_1 > \mu_2$, which is not necessarily the case in 15, and furthermore in 15 $c_0^{\mu_2 - 1} = 1$, but the only limitation on c_0 in 13 is $c_0^2 \neq 1$. Similarly it can be shown that the thirty types are all distinct.

Invariants of Differential Geometry by the Use of Vector Forms.

By C. D. Rice.

I. *Introduction.*

Let the equation of the surface be given by

$$x = f(uv), \tag{1}$$

where u and v are scalar variables. Partial derivatives in what follows with respect to u and v are represented by the subscripts 1 and 2 respectively. Derivatives with respect to s, the length of a curve on the surface, will be denoted by primes. The well-known constants in a point are given by

$$\left.\begin{array}{lll} E = -Sx_1x_1, & F = -Sx_1x_2, & G = -Sx_2x_2, \\ L = Sa_1x_1 = -Sax_{11}, & M = Sa_1x_2 = Sa_2x_1 = -Sax_{12}, & N = Sa_2x_2 = -Sax_{22}, \end{array}\right\} \tag{2}$$

where a is the unit vector normal to the surface at the point x. The vectors x', x_1, x_2, a', a_1, a_2 are all parallel to the tangent plane at the point. When a curve upon the surface is determined by a scalar relation

$$\phi(uv) = c,$$

we have

$$\phi_1 u' + \phi_2 v' = 0,$$

or

$$\frac{\phi_2}{u'} = -\frac{\phi_1}{v'} = r. \tag{3}$$

By the use of these relations we find from

$$x' = x_1 u' + x_2 v'$$

the relation

$$rx' = \phi_2 x_1 - \phi_1 x_2 = w. \tag{4}$$

In like manner we find

$$ra' = \phi_2 a_1 - \phi_1 a_2 = \overline{\omega}. \tag{5}$$

From the study of surfaces we have

$$\Delta^2 = EG - F^2 = Sx_1x_1Sx_2x_2 - Sx_1x_2Sx_1x_2 = -SVx_1x_2Vx_1x_2. \tag{6}$$

But we know that $\lambda a = Vx_1x_2$, and hence we find

$$\Delta a = Vx_1x_2. \tag{7}$$

By multiplication we find also

$$\Delta = -Sax_1x_2. \tag{8}$$

In like manner we write $\Delta_0 a = Va_1a_2$ and find

$$\Delta_0 = -Saa_1a_2. \tag{9}$$

But we know that the Gaussian curvature k is given by

$$\begin{aligned} \Delta k &= \frac{1}{\Delta}(LN - M^2) \\ &= \frac{1}{\Delta}(Sa_1x_1Sa_2x_2 - Sa_1x_2Sa_2x_1), \text{ since } Sa_2x_1 = Sa_1x_2 = M, \\ &= -\frac{1}{\Delta}SVa_1a_2Vx_1x_2 = -Saa_1a_2. \end{aligned} \tag{10}$$

$$\therefore \Delta k = \Delta_0. \tag{11}$$

By the use of r and w defined above we have the quadratic form:

$$\begin{aligned} \theta &= E\phi_2^2 - 2F\phi_2\phi_1 + G\phi_1^2 = -S(\phi_2 x_1 - \phi_1 x_2)(\phi_2 x_1 - \phi_1 x_2) = -Sww \\ &= -r^2 Sx'x', \text{ since } w = \phi_2 x_1 - \phi_1 x_2 = rx', \\ &= r^2, \text{ since } Sx'x' = -1. \end{aligned} \tag{12}$$

Also we have the quadratic:

$$\begin{aligned} \chi &= L\phi_2^2 - 2M\phi_2\phi_1 + N\phi_1^2 = S(\phi_2 x_1 - \phi_1 x_2)(\phi_2 a_1 - \phi_1 a_2) = Sw\overline{\omega} \\ &= r^2 Sa'x', \text{ since } ra' = \overline{\omega} \text{ and } rx' = w, \\ &= r^2\frac{1}{\rho}; \end{aligned} \tag{13}$$

where $\frac{1}{\rho} = Sa'x'$ is the curvature of a normal section through the tangent at the point.

Many expressions may be abbreviated by the use of the operator $\frac{d}{dn}$ where dn is an element of the arc of the curve on the surface at right angles to the given curve.

At any point P of the curve on the surface we have x', the unit vector along the tangent to the curve, and a the unit vector normal to the surface. Let us take ξ' a unit vector in the tangent plane at right angles to x. Then we have

$$\xi' = \frac{dx}{dn} = Vax'.$$

Let us write $\xi' = cx_1 + ex_2$ where c and e are scalars to be determined. We find

$$Sax_1\xi' = eSax_1x_2, \qquad Sax_2\xi' = cSax_2x_1,$$
$$\therefore\ \Delta e = -Sax_1\xi' = -Sax_1Vax' = -Sx_1x', \qquad \therefore\ \Delta c = Sax_2\xi' = Sax_2Vax' = Sx_2x'.$$

Hence we have

$$\xi' = -\frac{1}{\Delta}\{x_2Sx_1x' - x_1Sx_2x'\}, \text{ or } \frac{dx}{dn} = -\frac{1}{\Delta}\left\{\frac{\partial x}{\partial v}Sx_1x' - \frac{\partial x}{\partial u}Sx_2x'\right\}.$$

If now R be any function of u and v we find

$$\frac{dR}{dn} = \frac{dR}{dx}\cdot\frac{dx}{dn} = -\frac{1}{\Delta}\left\{\frac{\partial R}{\partial v}Sx_1x' - \frac{\partial R}{\partial u}Sx_2x'\right\}. \tag{14}$$

The operator $\frac{d}{dn}$ is very useful in what follows. In particular we have

$$\Delta\frac{d\phi}{dn} = -\{\phi_2Sx'x_1 - \phi_1Sx'x_2\} = -\{Sx'(\phi_2x_1 - \phi_1x_2)\} = -Sx'w = r,$$

$$\therefore\ \frac{d\phi}{dn} = \frac{r}{\Delta} = \beta, \tag{15}$$

where we define the quantity β by $\beta = \frac{r}{\Delta}$.

From a study of curves on surfaces we find

$$D = -Sax'x'' \tag{16}$$

to be the *Geodesic curvature*, and the expression

$$W = -Saa'x' \tag{17}$$

to be the *Geodesic torsion.*

Also we have the cubic

$$K = Sa''x' - Sa'x'' = Pu'^3 + 3Qu'^2v' + 3Ru'v'^2 + Sv'^3, \tag{18}$$

where

$$\left.\begin{aligned} P &= Sa_{11}x_1 - Sx_{11}a_1, \quad Q = Sa_{12}x_1 - Sx_{12}a_1 = Sa_{11}x_2 - Sx_{11}a_2, \\ R &= Sa_{12}x_2 - Sx_{12}a_2 = Sa_{22}x_1 - Sx_{22}a_1, \quad S = Sa_{22}x_2 - Sx_{22}a_2. \end{aligned}\right\} \tag{19}$$

The mean curvature h is given by

$$\begin{aligned} \Delta h &= \frac{1}{\Delta}\{EN - 2FM + GL\} \\ &= -\frac{1}{\Delta}\{(Sx_1x_1Sa_2x_2 - Sx_1x_2Sa_2x_1) - (Sx_1x_2Sa_1x_2 - Sx_2x_2Sa_1x_1)\} \\ &= \frac{1}{\Delta}\{SVx_1x_2Va_1x_2 - SVx_1x_2Va_2x_1\} = Saa_1x_2 - Saa_2x_1. \end{aligned} \tag{20}$$

From this expression for h we easily find

$$\left.\begin{aligned}\Delta^2 h_1 &= ER-2FQ+GP, \qquad \Delta^2 h_2 = ES-2FR+GQ,\\ \Delta^2 h' &= E(Ru'+Sv')-2F(Qu'+Rv')+G(Pu'+Qv').\end{aligned}\right\} \tag{21}$$

Change of Parameters.—Suppose we have the surface $\bar{x}=f(\bar{u}\bar{v})$ and wish to change to the parameters u, v, where $u=P(\bar{u}\bar{v})$, $v=Q(\bar{u}\bar{v})$. Let us write,

$$\frac{\partial u}{\partial \bar{u}}=\frac{\partial P}{\partial \bar{u}}=P_1, \quad \frac{\partial u}{\partial \bar{v}}=\frac{\partial P}{\partial \bar{v}}=P_2, \quad \frac{\partial v}{\partial \bar{u}}=\frac{\partial Q}{\partial \bar{u}}=Q_1, \quad \frac{\partial v}{\partial \bar{v}}=\frac{\partial Q}{\partial \bar{v}}=Q_2.$$

Then we have the transformation

$$\bar{x}_1=x_1P_1+x_2Q_1, \quad \bar{x}_2=x_1P_2+x_2Q_2. \tag{22}$$

The modulus of this transformation is denoted by

$$\delta=P_1Q_2-P_2Q_1=\begin{vmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{vmatrix}. \tag{23}$$

We wish now to consider forms of expression that are invariant in transformation. When the resulting expression has the same form multiplied by a power of δ it is said to be a *relative* invariant form. When it has the same form, but not multiplied by a power of δ it is said to be an *absolute* invariant.

Elementary Invariant Vector Forms.—When a vector $\bar{\xi}$ is transformed into the vector ξ we have

Form (I) $\quad d\bar{\xi}=\bar{\xi}_1 d\bar{u}+\bar{\xi}_2 d\bar{v}=(\xi_1P_1+\xi_2Q_1)\,d\bar{u}+(\xi_1P_2+\xi_2Q_2)\,d\bar{v}$
$\qquad\qquad =\xi_1(P_1d\bar{u}+P_2d\bar{v})+\xi_2(Q_1d\bar{u}+Q_2d\bar{v})=\xi_1du+\xi_2dv=d\xi.$

This shows the complete differential to be invariant, and hence we have the complete derivative to be invariant. Hence vector expressions composed of factors that are complete derivatives are invariant. Thus

$$D=-Sax'x'', \quad W=-Saa'x', \quad Sx'x', \text{ \&c.},$$

are invariant.

Form (II) $\quad V\bar{\xi}_1\bar{\xi}_2=V(\xi_1P_1+\xi_2Q_1)(\xi_1P_2+\xi_2Q_2)$
$\qquad\qquad =(P_1Q_2-P_2Q_1)V\xi_1\xi_2=\delta V\xi_1\xi_2,$

which shows this form to be invariant.

Under this form we have the invariants

$$\Delta=-Sax_1x_2=-SaVx_1x_2, \quad \Delta k=-Saa_1a_2=-SaVa_1a_2, \quad Saw_1w_2=SaVw_1w_2.$$

Form (III) $\quad S\bar{\xi}_1\bar{x}_2-S\bar{\xi}_2\bar{x}_1=S(\xi_1P_1+\xi_2Q_1)(x_1P_2+x_2Q_2)$
$\qquad -S(\xi_1P_2+\xi_2Q_2)(x_1P_1+x_2Q_1)=(P_1Q_2-P_2Q_1)(S\xi_1x_2-S\xi_2x_1)$
$\qquad =\delta(S\xi_1x_2-S\xi_2x_1).$

This form is made up of the difference of two scalar terms of which the partial derivatives 1, 2 in the first term are respectively 2, 1 in the second term.

In a similar manner we may show the 1, 2 and 2, 1 form to apply to a large number of invariant expressions. Thus,

$$Sx_1w_2-Sx_2w_1, \quad Sax_1w_2-Sax_2w_1, \quad x_1\Delta_2-x_2\Delta_1, \quad \phi_2x_1-\phi_1x_2, \text{ \&c.},$$

are all invariant forms.

Of especial importance under Form (III) is the invariant expression

$$\frac{dK}{dn}=-\frac{1}{\Delta}\{K_2Sx'x_1-K_1Sx'x_2\}, \quad \text{see (14)}$$

where K represents either a scalar or a vector quantity. By the use of this invariant expression we have, when K is a vector, a number of invariant forms. Thus each factor in

$$Sa\frac{dK}{dn}, \quad Sax'\frac{dK}{dn}, \quad Saw\frac{dK}{dn}, \quad Sx''\frac{dK}{dn}, \quad Sax'\frac{dK'}{dn}, \quad Saw\frac{dK'}{dn},$$

being invariant, the forms themselves are invariant.

Every invariant in differential geometry may be shown to be composed of one or more of the elementary forms (I), (II), or (III). If we can evaluate any of the invariant forms that we can write out from our simple elements in terms of well-known invariant expressions, we will be enabled to evaluate any invariant or covariant expression of differential geometry.

Forsyth has shown in his "Differential Geometry" that all invariants and covariants formed by the use of derivatives below the third order may be expressed in terms of any set of twelve such expressions that may be selected. From the set selected by him he derived eleven absolute invariant expressions as his fundamental set. For the set that we will use in this paper let us take the following by means of which we will evaluate all other invariant forms made by derivatives below the third order:

$D=-Sax'x''$, the geodesic curvature.

$W=-Saa'x'$, the geodesic torsion.

$k=-\frac{1}{\Delta}Saa_1a_2$, the Gaussian curvature where $\Delta=-Sax_1x_2$.

$\beta=\frac{r}{\Delta}=\frac{\sqrt{-Sww}}{\Delta}$, the differential parameter of the first order.

$h=\frac{1}{\Delta}\{Saa_1x_2-Saa_2x_1\}$, the mean curvature.

$\frac{1}{\rho}=Sa'x'=-Sax''$, the curvature of a normal section.

$$\frac{d\beta}{ds}, \; \frac{d\beta}{dn}, \; \frac{dh}{ds}, \; \frac{dh}{dn}, \; \frac{d}{ds}\left(\frac{1}{\rho}\right), \; \frac{d}{dn}\left(\frac{1}{\rho}\right).$$

Between these there exists the relation

$$k=\frac{h}{\rho}-\frac{1}{\rho^2}-W^2,$$

making in reality only eleven absolute invariant forms to be used.

We will now evaluate a number of *invariant forms* and will later show that invariants of differential geometry are composed of those forms. It will be seen that every form that we notice will be invariant by reason of elementary Form (I), Form (II) or Form (III). Hence, while we may evaluate invariant or covariant expressions of differential geometry by means of a set of eleven, we may express all such invariants or covariants in terms of *three elementary type forms.*

II. *Evaluation of Invariant Forms.*

We have defined

$$w=\phi_2x_1-\phi_1x_2=rx', \tag{24}$$

which is seen to be invariant by type form III. Also we have

$$Sww=r^2Sx'x'=-r^2, \text{ since } Sx'x'=-1. \tag{25}$$

$$Sx'w'=Sx'(r'x'+rx'')=-r', \text{ since } Sx'x''=0. \tag{26}$$

$$Sww'=rSx'w'=-rr'. \tag{27}$$

Since $Sax_1x_2=-\Delta$ we have

$$Sawx_2=Sa(\phi_2x_1-\phi_1x_2)x_2=-\phi_2\Delta, \quad Sawx_1=Sa(\phi_2x_1-\phi_1x_2)x_1=-\phi_1\Delta.$$

By differentiation we find

$$Saw_1x_2+Sawx_{12}=-\phi_{12}\Delta-\phi_2\Delta_1, \quad Saw_2x_1+Sawx_{12}=-\phi_{12}\Delta-\phi_1\Delta_2.$$

$$\therefore\ Saw_2x_1-Saw_1x_2=\phi_2\Delta_1-\phi_1\Delta_2=r\Delta'. \tag{28}$$

We have seen that we may write $ru'=\phi_2$ and $rv'=-\phi_1$, and from these we find $r_1u'+ru''=\phi_{12}=-r_2v'-rv''$,

$$\therefore\ r(u''+v'')=-\{r_1u'+r_2v'\}=-r'. \tag{29}$$

We have seen that

$$\Delta h=\ Saa_1x_2-Saa_2x_1. \text{ See (20)} \tag{30}$$

$$\Delta k=-Saa_1a_2. \text{ See (10).} \tag{31}$$

$$\begin{aligned}Sa'a'&=-\frac{1}{\Delta}Sa'a'Sax_1x_2\\ &=-\frac{1}{\Delta}\{Sa'x_1Saa'x_2-Sa'x_2Saa'x_1\}, \text{ since } Saa'=0\\ &=-\frac{1}{\Delta}\{Sa_1x'Saa'x_2-Sa_2x'Saa'x_1\}, \text{ since } Sa'x_1=Sa_1x', \text{ \&c.,}\end{aligned}$$

$$=-\frac{1}{\Delta}\{Sa'x'(Saa_1x_2-Saa_2x_1)+Sx'x_2Saa'a_1-Sx'x_1Saa'a_2\}$$
$$=-\frac{1}{\Delta}\left\{\frac{\Delta h}{\rho}-Saa_1a_2Sx'(x_2v'+x_1u')\right\},\ \text{since } Saa'a_1=v'Saa_2a_1$$
$$=-\frac{1}{\Delta}\left\{\frac{\Delta h}{\rho}+\Delta kSx'x'\right\}=-\frac{h}{\rho}+k. \tag{32}$$

It is easily seen that the vector $Va'x'$ is parallel to a, the unit vector normal to the surface. Hence we have

$$Wa=Va'x',\ \text{since } W=-Saa'x'.$$
$$\therefore\ W^2=-SVa'x'Va'x'=-Sa'x'Sa'x'+Sa'a'Sx'x'=-\frac{1}{\rho^2}+\frac{h}{\rho}-k. \tag{33}$$
$$WD=Saa'x'Sax'x''$$
$$=-\begin{vmatrix} Saa & Sax' & Sax'' \\ Sa'a & Sa'x' & Sa'x'' \\ Sx'a & Sx'x' & Sx'x'' \end{vmatrix}=-\begin{vmatrix} -1 & 0 & Sax'' \\ 0 & Sa'x' & Sa'x'' \\ 0 & -1 & 0 \end{vmatrix},\ \text{since } Sx'x''=0$$
$$=Sa'x''. \tag{34}$$

$$Saa'x''=-Sx'x'Saa'x''=-Sx'a'Sax'x''-Sx'x''Saa'x',\ \text{since } Sx'a=0$$
$$=\frac{D}{\rho},\ \text{since } Sx'x''=0. \tag{35}$$

$$Saa'w'=Saa'(r'x'+rx'')=-r'W+\frac{rD}{\rho}. \tag{36}$$

$$Sx'\frac{dx}{dn}=Sx'Vax'=0. \tag{37}$$

$$Sx'\frac{dw}{dn}=Sx'\left(r\frac{dx'}{dn}+x'\frac{dr}{dn}\right),\ \text{since } w=rx'$$
$$=-\frac{dr}{dn}. \tag{38}$$

$$Sx'\frac{da}{dn}=-\frac{1}{\Delta}\{Sx'x_1Sx'a_2-Sx'x_2Sx'a_1\},\ \text{see (14)}$$
$$=-\frac{1}{\Delta}\{Sx'x_1Sa'x_2-Sx'x_2Sa'x_1\},\ \text{since } Sa'x_2=Sx'a_2 \text{ and } Sa'x_1=Sx'a_1$$
$$=-\frac{1}{\Delta}SVx_1x_2Va'x'=-Saa'x',\ \text{since } \Delta a=Vx_1x_2$$
$$=W. \tag{39}$$

$$Sa\frac{dw}{dn}=-Sw\frac{da}{dn},\ \text{since } Saw=0$$
$$=-rSx'\frac{da}{dn}=-rW. \tag{40}$$

$$Sa'\frac{dx}{dn}= \quad Sa'Vax'=-Saa'x'=W. \tag{41}$$

$$\begin{aligned}Sax'\frac{dw}{dn}&=-\frac{1}{\Delta}\{Sx'x_1Sax'w_2-Sx'x_2Sax'w_1\}\\&=-\frac{1}{\Delta}\{Sx'x'(Sax_1w_2-Sax_2w_1)+Sx'w_2Sax'x_1-Sx'w_1Sax'x_2\}\\&=-\frac{1}{\Delta}\{r\Delta'+\Delta Sx'(w_2v'+w_1u')\}=-\frac{1}{\Delta}(r\Delta'+\Delta Sx'w')\\&=-\frac{1}{\Delta}(r\Delta'-\Delta r')=\frac{\Delta r'-r\Delta'}{\Delta}=\Delta\left(\frac{r}{\Delta}\right)'=\Delta\beta',\end{aligned}$$

since we define $\beta=\dfrac{r}{\Delta}$. (42)

$$Saw\frac{dw}{dn}=rSax'\frac{dw}{dn}=r\Delta\beta',\ \text{since } w=rx'. \tag{43}$$

$$\begin{aligned}Sax'\frac{dx'}{dn}&=Sax'\frac{d}{dn}\left(\frac{w}{r}\right)=\frac{1}{r}Sax'\frac{dw}{dn},\ \text{since } \frac{d}{du}\left(\frac{w}{r}\right)=\frac{1}{r}\frac{dw}{dn}+w\frac{d}{dn}\left(\frac{1}{r}\right)\\&=\frac{1}{r}\Delta\beta'=\frac{\beta'}{\beta}.\end{aligned} \tag{44}$$

$$Sax'\frac{dx}{dn}=SVax'Vax'=-SaaSx'x'=-1. \tag{45}$$

$$Saw'\frac{dx}{dn}=-Sx'x'Saw'\frac{dx}{dn}=-Sx'w'Sax'\frac{dx}{dn}=-r'. \tag{46}$$

$$\begin{aligned}Saw'\frac{dw}{dn}&=-Sx'x'Saw'\frac{dw}{dn}=-Sx'w'Sax'\frac{dw}{dn}-Sx'\frac{dw}{dn}Saw'x'\\&=+r'\Delta\beta'-rDSx'\frac{d}{dn}(rx'),\ \text{since } Sax'w'=Sax'(r'x'+rx'')=-rD\\&=+r'\Delta\beta'+rD\frac{dr}{dn}.\end{aligned} \tag{47}$$

$$Saw'\frac{dx'}{dn}=-Sx'x'Saw'\frac{dx'}{dn}=-Sx'w'Sax'\frac{dx'}{dn}-Sx'\frac{dx'}{dn}Saw'x'=\frac{\beta'}{\beta}\frac{r'}{1}. \tag{48}$$

$$Saa'\frac{dx'}{dn}=-Sx'x'Saa'\frac{dx'}{dn}=-Sx'a'Sax'\frac{dx'}{dn}-Sx'\frac{dx'}{dn}Saa'x'=-\frac{1}{\rho}\frac{\beta'}{\beta}. \tag{49}$$

$$\begin{aligned}Saa'\frac{dw}{dn}&=-Sx'x'Saa'\frac{dw}{dn}\\&=-Sx'a'Sax'\frac{dw}{dn}-Sx'\frac{dw}{dn}Saa'x'=-\frac{1}{\rho}\Delta\beta'-W\frac{dr}{dn}.\end{aligned} \tag{50}$$

$$Saa'\frac{dx}{dn}=SVaa'Vax'=-SaaSa'x'=\frac{1}{\rho}. \tag{51}$$

$$Sax'\frac{da}{dn}=-\frac{1}{\Delta}\{Sx'x_1Sax'a_2-Sx'x_2Sax'a_1\}$$

$$=-\frac{1}{\Delta}\{Sx'x'(Sax_1a_2-Sax_2a_1)+Sx'a_2Sax'x_1-Sx'a_1Sax'x_2\}$$

$$=-\frac{1}{\Delta}\{-\Delta h+\Delta Sx'(a_2v'+a_1u')\}=h-\frac{1}{\rho}. \tag{52}$$

$$Saw'\frac{da}{dn}=-Sx'x'Saw'\frac{da}{dn}$$

$$=-Sx'w'Sax'\frac{da}{dn}-Sx'\frac{da}{dn}Saw'x'=r'\left(h-\frac{1}{\rho}\right)-rDW. \tag{53}$$

$$Saa'\frac{da}{dn}=-Sx'x'Saa'\frac{da}{dn}=-Sx'a'Sax'\frac{da}{dn}-Sx'\frac{da}{dn}Saa'x'$$

$$=-\frac{1}{\rho}\left(h-\frac{1}{\rho}\right)+W^2=-k.\quad \text{See (33)}. \tag{54}$$

$$Sax''\frac{da}{dn}=-Sx'x'Sax''\frac{da}{dn}=-Sx'\frac{da}{dn}Sax''x'=-WD. \tag{55}$$

$$Saw_1w_2=-Sx'x'Saw_1w_2=-Sx'(x_1u'+x_2v')Saw_1w_2$$

$$=-\{Sx'x_1Saw'w_2-Sx'x_2Saw'w_1\}$$

$$=\Delta Saw'\frac{dw}{dn}=\Delta^2r'\beta'+\Delta rD\frac{dr}{dn}.\quad \text{See (47)}. \tag{56}$$

$$Sx''\frac{da}{dn}=-\frac{1}{W}Sx''\frac{da}{dn}Saa'x'$$

$$=+\frac{1}{W}Sx''a'Sax'\frac{da}{dn},\ \text{since}\ Sa'x'\frac{da}{dn}=0\ \text{and}\ Sx''x'=0$$

$$=D\left(h-\frac{1}{\rho}\right).\quad \text{See (34)}. \tag{57}$$

$$Sa'\frac{dw}{dn}=-\frac{1}{rD}Sa'\frac{dw}{dn}Sax'w'=-\frac{1}{rD}\left\{-Sa'x'Saw'\frac{dw}{dn}+Sa'w'Sax'\frac{dw}{dn}\right\}$$

$$=-\frac{1}{rD}\left\{-\frac{1}{\rho}\left(r'\Delta\beta'+rD\frac{dr}{du}\right)+\Delta\beta'\left(\frac{r'}{\rho}+rDW\right)\right\}$$

$$=-\frac{1}{rD}\left\{-rD\frac{dr}{du}+rDW\Delta\beta'\right\}=\frac{dr}{du}-\Delta W\beta'. \tag{58}$$

$$Sa'\frac{dx'}{dn}=-\frac{1}{rD}Sa'\frac{dx'}{dn}Sax'w'=-\frac{1}{rD}\left\{-Sa'x'Saw'\frac{dx'}{dn}+Sa'w'Sax'\frac{dx'}{dn}\right\}$$

$$=-\frac{1}{rD}\left\{-\frac{1}{\rho}\frac{\beta'}{\beta}r'+\frac{\beta'}{\beta}\left(\frac{r'}{\rho}+rWD\right)\right\}=-W\frac{\beta'}{\beta}. \tag{59}$$

$$Sw\frac{da'}{dn}=\frac{d}{dn}\left(\frac{r}{\rho}\right)-Sa'\frac{dw}{dn},\ \text{since}\ Sa'w=\frac{r}{\rho}$$
$$=\frac{d}{dn}\left(\frac{r}{\rho}\right)-Sa'\left(r\frac{dx'}{dn}+x'\frac{dr}{dn}\right),\ \text{since}\ w=rx'$$
$$=\frac{d}{dn}\left(\frac{r}{\rho}\right)+rW\frac{\beta'}{\beta}-\frac{1}{\rho}\frac{dr}{dn}$$
$$=r\frac{d}{dn}\left(\frac{1}{\rho}\right)+\frac{1}{\rho}\frac{dr}{dn}+rW\frac{\beta'}{\beta}-\frac{1}{\rho}\frac{dr}{dn}=r\frac{d}{dn}\left(\frac{1}{\rho}\right)+rW\frac{\beta'}{\beta}. \tag{60}$$

$$Sx'\frac{da'}{dn}=\frac{d}{dn}\left(\frac{1}{\rho}\right)+\frac{\Delta W\beta'}{r}. \tag{61}$$

$$Sx''\frac{dw}{dn}=-\frac{1}{W}Sx''\frac{dw}{dn}Saa'x'=-\frac{1}{W}\left\{Sx''aSa'x'\frac{dw}{dn}-Sx''a'Sax'\frac{dw}{dn}\right\}$$
$$=-\frac{1}{W}\left\{-\frac{1}{\rho}WSa\frac{dw}{dn}-WD\Delta\beta'\right\},\ \text{since}\ Wa=Va'x'$$
$$=-\frac{r}{\rho}W+D\Delta\beta'. \tag{62}$$

$$Sa''\frac{dx}{dn}=-Saa''x',\ \text{since}\ \frac{dx}{dn}=Vax'$$
$$=-Saa''x'-Saa'x''+Saa'x''=W'+\frac{D}{\rho}.\quad \text{See (35).} \tag{63}$$

$$Sa''\frac{dx}{dn}=\frac{1}{\Delta}\{Sx'x_2Sa''x_1-Sx'x_1Sa''x_2\}$$
$$=\frac{1}{\Delta}\{Sx'x_2(Sa''x_1-Sx''a_1)-Sx'x_1(Sa''x_2-Sx''a_2)\}+Sx''\frac{da}{dn}$$
$$=-\frac{1}{\Delta}[Sx'x_2\{Sa'(x_1)'-Sx'(a_1)'\}-Sx'x_1\{Sa'(x_2)'-Sx'(a_2)'\}]$$
$$+Sx''\frac{da}{dn},\ \text{since}\ Sx'a_1=Sa'x_1$$
$$=-\frac{1}{\Delta}[Sx'x_2\{Sa'(x')_1-Sx'(a')_1\}-Sx'x_1\{Sa'(x')_2-Sx'(a')_2\}]$$
$$+Sx''\frac{da}{dn}$$
$$=\frac{1}{\Delta}[Sx'x_2\{Sa'(x')_1+Sx'(a')_1\}-Sx'x_1\{Sa'(x')_2+Sx'(a')_2\}]$$
$$-\frac{2}{\Delta}[Sx'x_2Sa'(x')_1-Sx'x_1Sa'(x')_2]+Sx''\frac{da}{dn}=\frac{d}{dn}\left(\frac{1}{\rho}\right)$$
$$-\frac{2}{\Delta}Sa'\frac{dx'}{dn}+Sx''\frac{da}{dn}=\frac{d}{dn}\left(\frac{1}{\rho}\right)+2W\frac{\beta'}{\beta}+D\left(h-\frac{1}{\rho}\right). \tag{64}$$

See (59) and (57).

$$W' = -Saa''x' - Saa'x'' = Sa''Vax' - Saa'x'' = Sa''\frac{dx}{dn} - Saa'x''$$
$$= \frac{d}{dn}\left(\frac{1}{\rho}\right) + 2W\frac{\beta'}{\beta} + D\left(h - \frac{1}{\rho}\right) - \frac{D}{\rho}. \quad \text{See (64) and (35).}$$
$$= \frac{d}{dn}\left(\frac{1}{\rho}\right) + 2W\frac{\beta'}{\beta} + D\left(h - \frac{2}{\rho}\right). \tag{65}$$

$$Saa'a'' = -Sx'x'Saa'a'' = -Sx'a'Sax'a'' - Sx'a''Saa'x'$$
$$= -\frac{1}{\rho}(Sax''a' + Sax'a'') + \frac{1}{\rho}Sax''a' + WSa''x'$$
$$= -\frac{W'}{\rho} - \frac{D}{\rho^2} + W(K + WD), \text{ since } K = Sa''x' - Sa'x'' = Sa''x' - WD$$
$$= -\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right) - D\left(\frac{h}{\rho} - \frac{1}{\rho^2} - W^2\right) - \frac{2}{\rho}\frac{W\beta'}{\beta} + WK$$
$$= -\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right) - Dk - \frac{2W\beta'}{\beta} + WK. \tag{66}$$

By the use of similar methods we find

$$Sx_2w_1 - Sx_1w_2 = -\Delta\left(rD - \frac{dr}{dn}\right). \tag{67}$$

$$Sx'(\Delta_1a_2 - \Delta_2a_1) = \Delta\left(W\Delta' - \frac{1}{\rho}\frac{d\Delta}{dn}\right). \tag{68}$$

$$Saa_1w_2 - Saa_2w_1 = -\Delta W\left(rD + \frac{dr}{dn}\right) - \frac{\Delta}{\rho}\Delta'\beta - \frac{2\Delta^2\beta'}{\rho} + r'\Delta h. \tag{69}$$

$$Sax'(\Delta_1w_2 - \Delta_2w_1) = -r\Delta'^2 + \Delta rD\frac{d\Delta}{dn} + \Delta\Delta'r'. \tag{70}$$

$$Sax'(\Delta_1a_2 - \Delta_2a_1) = \Delta\Delta'h - \Delta W\frac{d\Delta}{dn} - \frac{\Delta}{\rho}\Delta'. \tag{71}$$

$$Sa_1w_2 - Sa_2w_1 = -\Delta rD\left(h - \frac{1}{\rho}\right) + \frac{\Delta}{\rho}\frac{dr}{dn} - 2\Delta^2W\beta' - r\Delta'W. \tag{72}$$

$$Saa_1(x')_2 - Saa_2(x')_1 = -\frac{\Delta\beta'}{\rho\beta} - \Delta WD. \tag{73}$$

$$Sa(a')_1x_2 - Sa(a')_2x_1 = \Delta h' - \frac{\Delta\beta'}{\beta}\left(h - \frac{1}{\rho}\right) + \Delta WD. \tag{74}$$

Each form evaluated above is seen to be invariant by virtue of the type forms I, II or III, of which it is composed.

III. *Invariants of Differential Geometry.*

In what follows invariants or covariants of differential geometry will be seen to be composed of one or more invariant forms, and hence may be expressed in terms of one or more of the three type forms I, II and III. It will thus be seen, also, that the invariants and covariants may be evaluated by the use of the results obtained in the previous section.

In making the evaluations of invariant forms the derivatives of r and Δ were used in several instances. The reader will notice that in combining two or more of these forms to express an invariant or covariant in differential geometry that the derivatives of r and Δ are eliminated by the use of the relation $r=\Delta\beta$.

In what follows the invariancy of an expression will be indicated by giving at the right the type form or forms of which it is composed.

The quadratic form

$$\theta=E\phi_2^2-2F\phi_2\phi_1+G\phi_1^2$$

was shown in (25) to be given by

$$\theta=-Sww=-S(\phi_2x_1-\phi_1x_2)(\phi_2x_1-\phi_1x_2) \quad \text{(Form III)}. \qquad \text{(a)}$$

From this we obtain the absolute invariant

$$\frac{\theta}{\Delta^2}=-\frac{Sww}{\Delta^2}=\beta^2, \qquad \text{(a')}$$

where β^2 is the differential parameter of the first order.

Again we have the quadratic form

$$\begin{aligned}\chi&=L\phi_2^2-2M\phi_2\phi_1+N\phi_1^2=S(\phi_2x_1-\phi_1x_2)(\phi_2a_1-\phi_1a_2) \quad \text{(Form III)}.\\ &=Sw\overline{\omega}, \text{ since } \overline{\omega}=\phi_2a_1-\phi_1a_2\\ &=r^2Sx'a', \text{ since } w=rx' \text{ and } \overline{\omega}=ra'=\frac{r^2}{\rho}. \quad \text{See (13)}. \qquad \text{(b)}\end{aligned}$$

From this we obtain the absolute invariant form

$$\frac{\chi}{\Delta^2}=\frac{\beta^2}{\rho}, \text{ since } \beta=\frac{r}{\Delta}. \qquad \text{(b')}$$

We have seen also in (10),

$$\begin{aligned}k&=\frac{1}{\Delta^2}(LN-M^2)=\frac{1}{\Delta^2}(Sa_1x_1Sa_2x_1-Sa_2x_1Sa_1x_2)\\ &=-\frac{1}{\Delta^2}SVa_1a_2Vx_1x_2 \quad \text{(Form II)}\\ &=-\frac{1}{\Delta}Saa_1a_2. \qquad \text{(c)}\end{aligned}$$

In differential geometry this absolute invariant is often written

$$k=\frac{1}{\rho_1\rho_2}. \tag{c'}$$

After omitting the numerical factors the functional determinant of θ and χ may be written

$$J_{\theta\chi}=-\begin{vmatrix} Sx_1w, & Sx_2w \\ Sa_1w, & Sa_2w \end{vmatrix}=-r^2\{Sx_1x'Sx'a_2-Sx_2x'Sx'a_1\}$$

$$=\ \Delta r^2Sx'\frac{da}{dn}=\Delta r^2W. \quad \text{See (39).} \tag{d}$$

From this we find the absolute invariant

$$\frac{J_{\theta\chi}}{\Delta^3}=\beta^2W. \tag{d'}$$

We have the quadratic

$$d=A\phi_2^2-2B\phi_2\phi_1+C\phi_1^2=r^2(Au'^2+2Bu'v'+Cv^2),$$

where

$$\Delta A=\Delta\phi_{11}+Sawx_{11}, \quad \Delta B=\Delta\phi_{12}+Sawx_{12}, \quad \Delta C=\Delta\phi_{22}+Sawx_{22}.$$

$$\therefore\ d=\frac{r^2}{\Delta}\{\Delta(\phi_{11}u'^2+2\phi_{12}u'v'+\phi_{22}v'^2)+Saw(x_{11}u'^2+2x_{12}u'v'+x_{22}v'^2)\}$$

$$=\frac{r^2}{\Delta}\{-\Delta(\phi_1u''+\phi_2v'')+Saw(x''-x_1u''-x_2v'')\}$$

$$=\frac{r^2}{\Delta}\{-\Delta(\phi_1u''+\phi_2v'')+Sawx''+\Delta(\phi_1u''+\phi_2v'')\}$$

$$=\frac{r^3}{\Delta}Sax'x'' \quad \text{(Form I)}. \tag{e}$$

In this last form d is seen to be invariant. We have also the absolute invariant form

$$\frac{d}{\Delta^2}=-\beta^3D, \text{ since } D=-Sax'x''. \tag{e'}$$

More useful expressions for A, B and C may be found as follows:

$$Sawx_1=Sa(\phi_2x_1-\phi_1x_2)x_1=-\phi_1\Delta, \quad Sawx_2=Sa(\phi_2x_1-\phi_1x_2)x_2=-\phi_2\Delta.$$

By differentiation we have,

$$Sawx_{11}+Saw_1x_1=-\phi_1\Delta_1-\phi_{11}\Delta, \quad Sawx_{12}+Saw_2x_1=-\phi_1\Delta_2-\phi_{12}\Delta,$$
$$Sawx_{12}+Saw_1x_2=-\phi_2\Delta_1-\phi_{12}\Delta, \quad Sawx_{22}+Saw_2x_2=-\phi_2\Delta_2-\phi_{22}\Delta,$$

and by the use of these relations we find

$$\Delta A=-\phi_1\Delta_1-Saw_1x_1, \quad \Delta B=-\phi_1\Delta_2-Saw_2x_1=-\phi_2\Delta_1-Saw_1x_2,$$
$$\Delta C=-\phi_2\Delta_2-Saw_2x_2.$$

The intermediate invariant of θ and d may be written

$$I=AG-2BF+CE=\frac{1}{\Delta}\{Sx_2x_2(\Delta_1\phi_1+Saw_1x_1)-Sx_1x_2(\phi_2\Delta_1+Saw_1x_2)$$
$$-Sx_1x_2(\phi_1\Delta_2+Saw_2x_1)+Sx_1x_1(\phi_2\Delta_2+Saw_2x_2)\}$$
$$=\frac{1}{\Delta}\{\Delta_1Sx_2(\phi_1x_2-\phi_2x_1)-\Delta_2Sx_1(\phi_1x_2-\phi_2x_1)+Sx_2w_1Sax_2x_1$$
$$+Sx_2x_1Saw_1x_2-Sx_1x_2Saw_1x_2-Sx_1w_2Sax_2x_1+Sx_1x_1Saw_2x_2-Sx_1x_1Saw_2x_2\}$$
$$=\frac{1}{\Delta}\{\Delta_2Sx_1w-\Delta_1Sx_2w+\Delta(Sx_2w_1-Sx_1w_2)\}\quad\text{(Form III)}$$
$$=\frac{1}{\Delta}\left\{-r\Delta\frac{d\Delta}{dn}-\Delta^2\left(rD-\frac{dr}{dn}\right)\right\}.\quad\text{See (14) and (67)}$$
$$=-\Delta rD+\Delta\frac{dr}{dn}-\beta\frac{d\Delta}{dn}\Delta=\Delta^2\frac{d\beta}{dn}-\beta\Delta^2D,\text{ since } r=\Delta\beta$$
$$=\Delta^2\left(\frac{d\beta}{dn}-\beta D\right),\tag{f}$$

and from this we have the absolute invariant

$$\frac{I}{\Delta^2}=\frac{d\beta}{dn}-\beta D.\tag{f'}$$

It is well to notice that the above invariant may be written

$$I=\Delta\left\{Sx_2\left(\frac{w}{\Delta}\right)_1-Sx_1\left(\frac{w}{\Delta}\right)_2\right\}\quad\text{(Form III)}$$

or $$\frac{I}{\Delta^2}=\frac{1}{\Delta}\left\{\left(\frac{Sx_2w}{\Delta}\right)_1-\left(\frac{Sx_1w}{\Delta}\right)_2\right\}=\frac{1}{\Delta}\left\{\frac{\partial}{\partial u}\left(\frac{G\phi_1-F\phi_2}{\Delta}\right)-\frac{\partial}{\partial v}\left(\frac{E\phi_2-F\phi_1}{\Delta}\right)\right\},$$

which is the *differential parameter of the second order.*

By omitting the numerical factors, the Jacobian of θ and d may be written:

$$J_{\theta d}=-\begin{vmatrix}Sx_1w, & Sx_2w\\ (A\phi_2-B\phi_1), & (B\phi_2-C\phi_1)\end{vmatrix}$$
$$=\frac{1}{\Delta}\begin{vmatrix}Sx_1w, & \phi_2(\phi_1\Delta_1+Saw_1x_1)-\phi_1(\phi_2\Delta_1+Saw_1x_2),\\ Sx_2w, & \phi_2(\phi_1\Delta_2+Saw_2x_1)-\phi_1(\phi_2\Delta_2+Saw_2x_2)\end{vmatrix}$$
$$=\frac{1}{\Delta}\begin{vmatrix}Sx_1w & Sx_2w\\ Saw_1w & Saw_2w\end{vmatrix},\text{ since } w=\phi_2x_1-\phi_1x_2$$
$$=-\frac{r^2}{\Delta}\{Sx_1x'Sax'w_2-Sx_2x'Sax'w_1\}\quad\text{(Form III)}$$
$$=r^2Sax'\frac{dw}{du}=r^2\Delta\beta'.\quad\text{See (42).}\tag{g}$$
$$\therefore\ \frac{J\theta d}{\Delta^3}=\beta^2\beta'.\tag{g'}$$

The cubic

$$\overline{K}=P\phi_2^3-3Q\phi_2^2\phi_1+3R\phi_2\phi_1^2-S\phi_1^3=r^3(Pu'^3+3Qu'^2v'+3Ru'v'^2+Sv'^3)$$
$$=r^3(Sa''x'-Sa'x'')\quad\text{(Form I).}\tag{h}$$

is seen in this last form to be invariant.

This cubic may be expressed in terms of well-known invariants as follows:

$$\begin{aligned}\overline{K}&=r^3(Sa''x'+Sa'x''-2Sa'x'')\\&=r^3\left\{\frac{d}{ds}(Sa'x')-2Sa'x''\right\}=r^3\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-2WD\right\},\end{aligned}$$

and from this we find

$$\frac{\overline{K}}{\Delta^3}=\beta^3\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-2WD\right\}. \tag{h$'$}$$

From the cubic $\overline{K}$, let us take

$$\begin{aligned}K=Sa''x'-Sx''a'&=S(a_{11}u'^2+2a_{12}u'v'+a_{22}v'^2)x'-S(x_{11}u'^2+2x_{12}u'v'+x_{22}v'^2)a'\\&=u'^3\{Sa_{11}x_1-Sx_{11}a_1\}+u'^2v'\{(Sa_{11}x_2-Sx_{11}a_2)+2(Sa_{12}x_1-Sx_{12}a_1)\}\\&\quad+u'v'^2\{2(Sa_{12}x_2-Sx_{12}a_2)+(Sa_{22}x_1-Sx_{22}a_1)\}+v'^3\{Sa_{22}x_2-Sx_{22}a_2\}.\end{aligned}$$

Now we have $Sa_1x_2=Sx_1a_2$, and by differentiation we find

$$Sa_{11}x_2-Sx_{11}a_2=Sa_{12}x_1-Sx_{12}a_1,\quad Sa_{12}x_2-Sx_{12}a_2=Sa_{22}x_1-Sx_{22}a_1.$$

If now we write

$$K=Pu'^3+3Qu'^2v'+3Ru'v'^2+Sv'^3,$$

we have from the above,

$$\begin{aligned}&P=Sa_{11}x_1-Sx_{11}a_1,\quad Q=Sa_{12}x_1-Sx_{12}a_1=Sa_{11}x_2-Sx_{11}a_2,\\&R=Sa_{12}x_2-Sx_{12}a_2=Sa_{22}x_1-Sx_{22}a_1,\quad S=Sa_{22}x_2-Sx_{22}a_2.\end{aligned}$$

Let us now denote

$$K_1=Pu'^2+2Qu'v'+Rv'^2,\quad K_2=Qu'^2+2Ru'v'+Sv'^2.$$

Then we find

$$\begin{aligned}K_1&=u'(Pu'+Qv')+v'(Qu'+Rv')\\&=u'\{S(a_1)'x_1-S(x_1)'a_1\}+v'\{S(a_2)'x_1-S(x_2)'a_1\}\\&=u'\{S(a')_1x_1-S(x')_1a_1\}+v'\{S(a')_2x_1-S(x')_2a_1\}\\&=Sx_1\{(a')_1u'+(a')_2v'\}-Sa_1\{(x')_1u'+(x')_2v'\}\\&=Sa''x_1-Sx''a_1,\end{aligned}$$

and in like manner $$K_2=Sa''x_2-Sx''a_2.$$

The expression

$$\begin{aligned}\sigma=&\ \{E^2S-3EFR+(EG+2F^2)Q-FGP\}\phi_2\\&\quad-\{EFS-(EG+2F^2)R+3FGQ-G^2P\}\phi_1\\=&\ \{E(ES-2FR+GQ)-F(ER-2FQ+GP)\}\phi_2\\&\quad-\{F(ES-2FR+GQ)-G(ER-2FQ+GP)\}\phi_1\\=&\ \Delta^2\{(Eh_2-Fh_1)\phi_2-(Fh_2-Gh_1)\phi_1\}\quad\text{See (21)}\\=&-\Delta^2\{h_2Sx_1(\phi_2x_1-\phi_1x_2)-h_1Sx_2(\phi_2x_1-\phi_1x_2)\}\\=&-\Delta^2r\{h_2Sx_1x'-h_1Sx_2x'\}\quad\text{(Form III)}\\=&\ \Delta^3r\frac{dh}{dn},\end{aligned} \tag{i}$$

is seen in the last two forms to be invariant.

From this we easily derive the absolute invariant form,

$$\frac{\sigma}{\Delta^4}=\beta\frac{dh}{dn}. \tag{i'}$$

Also we have

$$\begin{aligned}\sigma'&=(ER-2FQ-GP)\phi_2-(ES-2FR+GQ)\phi_1\\&=\Delta^2(h_1\phi_2-h_2\phi_1)\quad\text{See (21)}\quad\text{(Form III)}\\&=r\Delta^2h'\end{aligned} \tag{j}$$

which is seen to be invariant. It may also be written

$$\frac{\sigma'}{\Delta^3}=\beta h', \tag{j'}$$

as an absolute invariant.

The Jacobian of the cubic K and the quadratic θ may be written:

$$J_{K\theta}=-r^3\begin{vmatrix}Sx'x_1 & Sx'x_2\\ Sa''x_1-Sx''a_1 & Sa''x_2-Sx''a_2\end{vmatrix}=\Delta r^3\left\{Sa''\frac{dx}{dn}-Sx''\frac{da}{dn}\right\}$$

(Forms I and III)

$$=\Delta r^3\left\{\frac{d}{dn}\left(\frac{1}{\rho}\right)+2\frac{W\Delta\beta'}{r}+D\left(h-\frac{1}{\rho}\right)-D\left(h-\frac{1}{\rho}\right)\right\}$$

See (57) and (64)

$$=\Delta r^3\left\{\frac{d}{dn}\left(\frac{1}{\rho}\right)+2\frac{W\beta'}{\beta}\right\}. \tag{k}$$

From this we have the absolute invariant form,

$$\frac{J_{K\theta}}{\Delta^4}=\beta^3\frac{d}{dn}\left(\frac{1}{\rho}\right)+2W\beta'\beta^2. \tag{k'}$$

As a special case of the quadratic

$$-Sx'x'=Eu'^2+2Fu'v'+Gv'^2$$

and the cubic $\quad K=Pu'^3+3Qu'^2v'+3Ru'v'^2+Sv'^3=Sa''x'-Sx''a',$

we may write $\quad J_{23}=\dfrac{\Delta}{\beta}\left\{\beta\dfrac{d}{dn}\left(\dfrac{1}{\rho}\right)+2W\beta'\right\}.$

The Jacobian of P and χ may be written

$$\begin{aligned}J_{P\chi}&=6r^3\begin{vmatrix}-Sa_1x' & Sa_2x'\\ (Sa''x_1-Sx''a)_1- & (Sa''x_2-Sx''a_2)\end{vmatrix}\\&=6r^3\begin{vmatrix}Sa'x_1 & Sa'x_2\\ Sa''x_1 & Sa''x_2\end{vmatrix}-6r^3\begin{vmatrix}Sa_1x' & Sa_2x'\\ Sx''a_1 & Sx''a_2\end{vmatrix},\ \text{since } Sa_1x'=Sa'x_1\\&=-6r^3\{SVx_1x_2Va'a''-SVa_1a_2Vx'x''\}=-6\Delta r^3\{Saa'a''-kSax'x''\}\quad\text{(Form I)}\\&=+6\Delta r^3\left\{\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right)+Dk+\frac{2W\beta'}{\rho\beta}-WP-Dk\right\}\quad\text{See (66) and (16)}\\&=6\Delta^3\left\{\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right)+\frac{2W\beta'}{\rho\beta}-WP\right\}.\end{aligned} \tag{l}$$

$$\therefore\ \frac{J_{P\chi}}{\Delta^4}=6\beta^3\left\{\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right)+\frac{2W\beta'}{\rho\beta}-WP\right\}. \tag{l'}$$

The discriminant of the quadratic d may be given

$$\begin{aligned}
\Delta^2(AC-B^2) &= (\phi_1\Delta_1+Saw_1x_1)(\phi_2\Delta_2+Saw_2x_2)-(\phi_1\Delta_2+Saw_2x_1)(\phi_2\Delta_1+Saw_1x_2)\\
&= \Delta_1Saw_2(\phi_1x_2-\phi_2x_1)+\Delta_2Saw_1(\phi_2x_1-\phi_1x_2)+Saw_1x_1Saw_2x_2\\
&\quad -Saw_2x_1Saw_1x_2=Saw(\Delta_1w_2-\Delta_2w_1)+Saw_1\{x_1Saw_2x_2-x_2Saw_2x_1\}\\
&= Saw(\Delta_1w_2-\Delta_2w_1)+Saw_1\{w_2Sax_1x_2+x_2Saw_2x_1-x_2Saw_2x_1\}\\
&= Saw(\Delta_1w_2-\Delta_2w_1)-\Delta Saw_1w_2 \quad \text{(Forms II and III)}\\
&= -r^2\Delta'^2+\Delta r^2D\frac{d\Delta}{du}+r\Delta\Delta'r'-\Delta^3r'\beta'-\Delta^2\frac{dr}{du}rD\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{See (70) and (56)}\\
&= -\Delta^2rD\left(\frac{dr}{du}-\beta\frac{d\Delta}{du}\right)-r\Delta\beta\Delta'^2+\Delta r\Delta'(\Delta\beta'+\Delta'\beta)-\Delta^3r'\beta'\\
&= -\Delta^2rD\left(\frac{dr}{du}-\beta\frac{d\Delta}{du}\right)+\Delta^3\beta'(\beta\Delta'-r')\\
&= -\Delta^2rD\left(\frac{dr}{dn}-\beta\frac{d\Delta}{dn}\right)-\Delta^4\beta'^2=-\Delta^3rD\frac{d\beta}{dn}-\Delta^4\beta'^2. \qquad \text{(m)}
\end{aligned}$$

$$\therefore\ \frac{1}{\Delta^2}(AC-B^2)=-\beta'^2-\beta D\frac{d\beta}{dn}. \qquad \text{(m')}$$

The intermediate invariant of d and χ is given by

$$\begin{aligned}
NA-2MB+LC &= -\frac{1}{\Delta}[\{Sa_2x_2(\Delta_1\phi_1+Saw_1x_1)-Sa_2x_1(\Delta_1\phi_2+Saw_1x_2)\}\\
&\quad -\{Sa_1x_2(\Delta_2\phi_1+Saw_2x_1)-Sa_1x_1(\Delta_2\phi_2+Saw_2x_2)\}]\\
&= -\frac{1}{\Delta}[\Delta_1Sa_2(\phi_1x_2-\phi_2x_1)-\Delta_2Sa_1(\phi_1x_2-\phi_2x_1)\\
&\quad +Sa_2x_2Saw_1x_1-Sa_2x_1Saw_1x_2-Sa_1x_2Saw_2x_1+Sa_1x_1Saw_2x_2]\\
&= -\frac{1}{\Delta}\{(\Delta_2Swa_1-\Delta_1Swa_2)+\Delta\{Sa_2w_1-Sa_1w_2\}\} \quad \text{(Form III)}\\
&= \beta\Delta\left(W\Delta'-\frac{1}{\rho}\frac{d\Delta}{dn}\right)-\Delta rD\left(h-\frac{1}{\rho}\right)+\frac{\Delta}{\rho}\frac{dr}{dn}\\
&\qquad -2W\Delta^2\beta'-\Delta'rW \quad \text{See (68) and (72)}\\
&= \frac{\Delta^2}{\rho}\frac{d\beta}{dn}-\Delta^2\beta D\left(h-\frac{1}{\rho}\right)-2W\Delta^2\beta'. \qquad \text{(n)}
\end{aligned}$$

$$\therefore\ \frac{1}{\Delta^2}\left\{NA-2MB+LC\right\}=\frac{1}{\rho}\frac{d\beta}{dn}-\beta D\left(h-\frac{1}{\rho}\right)-2W\beta'. \qquad \text{(n')}$$

It is well to notice that the above gives the form

$$NA-2MB+LC=\Delta\left\{Sa_1\left(\frac{w}{\Delta}\right)_2-Sa_2\left(\frac{w}{\Delta}\right)_1\right\} \quad \text{(Form III)}.$$

We have,

$$\begin{vmatrix} E & F & G \\ L & M & N \\ A & B & C \end{vmatrix} = C(EM-FL)+B(LG-FM)+B(FM-EN)+A(FN-MG).$$

Now we find

$$\begin{aligned}
EM-FL &= -[Sx_1x_1Sa_1x_2-Sx_1x_2Sa_1x_1] = -SVx_1x_2Va_1x_1 = -\Delta Saa_1x_1, \\
LG-FM &= -[Sx_2x_2Sa_1x_1-Sx_1x_2Sa_1x_2] = SVx_1x_2Va_1x_2 = \Delta Saa_1x_2, \\
FM-EN &= -[Sx_1x_2Sa_2x_1-Sx_1x_1Sa_2x_2] = SVx_1x_2Va_2x_1 = \Delta Saa_2x_1, \\
FN-MG &= -[Sx_1x_2Sa_2x_2-Sa_2x_1Sx_2x_2] = -SVx_1x_2Va_2x_2 = -\Delta Saa_2x_2.
\end{aligned}$$

$$\begin{aligned}
\therefore \begin{vmatrix} E & F & G \\ L & M & N \\ A & B & C \end{vmatrix} &= Saa_1\{x_1(\phi_2\Delta_2+Saw_2x_2)-x_2(\phi_1\Delta_2+Saw_2x_1)\} \\
&\quad -Saa_2\{x_1(\phi_2\Delta_1+Saw_1x_2)-x_2(\phi_1\Delta_1+Saw_1x_1)\} \\
&= \Delta_2Saa_1w-\Delta_1Saa_2w+Saa_1\{w_2Sax_1x_2+x_2Saw_2x_1-x_2Saw_2x_1\} \\
&\quad -Saa_2\{w_1Sax_1x_2+x_2Saw_1x_1-x_2Saw_2x_1\} \\
&= Saw(\Delta_1a_2-\Delta_2a_1)+\Delta(Saa_2w_1-Saa_1w_2) \quad \text{(Form III)} \\
&= r\Delta'\Delta h - r\Delta W\frac{d\Delta}{dn} - r\frac{\Delta}{\rho}\Delta' + \Delta^2W\left(rD+\frac{dr}{dn}\right) \\
&\qquad + \frac{\Delta^2}{\rho}\beta\Delta' + \frac{2\Delta^3\beta'}{\rho} - r'\Delta^2h \quad \text{See (69) and (71)} \\
&= \Delta^2h(\beta\Delta'-r') - \Delta^2W\left(\beta\frac{d\Delta}{dn}-\frac{dr}{dn}\right) + \Delta^2WrD + \frac{2\Delta^3\beta'}{\rho} \\
&= -\Delta^3h\beta' + \Delta^3W\frac{d\beta}{dn} + \Delta^3\beta WD + \frac{2\Delta^3\beta'}{\rho} \\
&= \Delta^3\left[\left(\frac{2}{\rho}-h\right)\beta' + W\left\{\frac{d\beta}{dn}+\beta D\right\}\right]. \qquad \text{(o)}
\end{aligned}$$

$$\therefore \frac{1}{\Delta^3}\begin{vmatrix} E & F & G \\ L & M & N \\ A & B & C \end{vmatrix} = \left(\frac{2}{\rho}-h\right)\beta' + W\left(\frac{d\beta}{dn}+\beta D\right). \qquad \text{(o')}$$

Many other examples could be given, but enough has been shown to illustrate the use of vector methods in discussing certain forms found in differential geometry. The reader will see how much more direct and simple these expressions and reductions are in vector forms than in the use of the more tedious expressions in Cartesian coordinates.

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*On Certain Saltus Equations.**

BY HENRY BLUMBERG.

Introduction.

Instead of attempting to introduce into our discussion the utmost generality, we shall, for the sake of greater simplicity, confine ourselves to the consideration of a real, one-valued function $f(x)$, defined in the linear continuum, bounded at every point—and hence, according to the Borel theorem on sets of intervals, in every interval—and unrestricted as to continuity. From $f(x)$ we derive three new functions: $u(f, x)$, the upper-bound ($=$maximum) function; $l(f, x)$, the lower-bound ($=$minimum) function; and $s(f, x)=u(f, x) -l(f, x)$, the saltus ($=$oscillation) function of f.† Just as the notion of derivative at once suggests that of differential equation, so the notion of saltus leads to that of "saltus equation." It is the principal object of the present paper to give what may be regarded as complete solutions of such saltus equations in several simple cases. For the sake of greater brevity, we write as follows the successive saltus functions derived from $f(x)$:

$$s(f, x)=s'_f(x), \quad s(s'_f, x)=s''_f(x), \quad s(s''_f, x)=s'''_f(x), \ldots .$$

Because of a theorem due to Sierpiński, ‡ which asserts that $s'''_f(x)=s''_f(x)$ no matter what function $f(x)$ we start with, there is no need of considering saltus equations beyond the "second order," *i. e.*, equations involving a saltus with an index greater than 2. The saltus equations $s'_f(x)=g(x)f(x)$ and $s''_f(x)=g(x)s'_f(x)$, where $g(x)$ is an arbitrarily given continuous function and $f(x)$ is sought, are among those above referred to as "completely solved" in this paper.

The first section deals with several properties of functions of interest in themselves and useful later. Section 2 deals with the equation $s'_f(x)=g(x)f(x)$; Section 3, with the equation $s''_f(x)=g(x)s'_f(x)$; and the appendix indicates several lines of generalization.

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* Read before the American Mathematical Society, December 26, 1913.

† For the definitions of these functions see the author's paper, "On Certain General Properties of Functions," *Annals of Mathematics*, Vol. XVIII (1917), p. 147, and Hobson, "The Theory of Functions of a Real Variable" (1907), Art. 180.

‡ *Bulletin de l'Académie des Sciences de Cracovie* (1910), pp. 633–634.

Section 1. Preliminary Theorems and Several Simple Saltus Equations.

Theorem I. *If two given functions are upper-semi-continuous (lower-semi-continuous) at a given point ξ, the saltus of their sum at ξ is greater than or equal to the saltus of each of the given functions at ξ.*

For let $f_1(x)$ and $f_2(x)$ be upper-semi-continuous at ξ. This condition is equivalent to the relations $f_1(\xi)=u(f_1,\xi)$ and $f_2(\xi)=u(f_2,\xi)$, where $u(f_1,x)$ and $u(f_2,x)$ are the upper-bound functions of f_1 and f_2, respectively. If $s'_{f_1}(\xi)=h$, a sequence $\{\xi_n\}$ of real numbers exists such that $\lim\limits_{n\to\infty}\xi_n=\xi$ and $\lim\limits_{n\to\infty}f_1(\xi_n)=u(f_1,\xi)-h=f_1(\xi)-h$. Moreover, since $f_2(\xi)$ is upper-semi-continuous, we have $\lim\limits_{n\to\infty}f_2(\xi_n)\leqq f_2(\xi)$. Hence

$$\lim_{n\to\infty}[f_1(\xi_n)+f_2(\xi_n)]\leqq f_1(\xi)+f_2(\xi)-h,$$

which shows that the function $f(x)=f_1(x)+f_2(x)$ has a saltus $\geqq h$ at ξ. In the same way, we prove the theorem for $f_2(x)$, and for two lower-semi-continuous functions.

Since the saltus function $s'_f(x)=u(f,x)-l(f,x)$ is actually the sum of two upper-semi-continuous functions, $u(f,x)$ and $-l(f,x)$, we have, as an application of Theorem I,

Theorem II. *The second saltus function is at every point greater than or equal to the saltus of both the upper-bound function and the lower-bound function.*

If in Theorem I it happens that the saltus of the sum is zero at ξ, then it follows that the saltus of both f_1 and f_2 is zero at ξ. That is, we have

Theorem III. *If the sum of two given upper-semi-continuous functions is continuous at a point ξ, then each of the given functions is continuous as ξ.**

Either from Theorem II or Theorem III, we obtain by specialization

Theorem IV. *If $s'_f(x)$ is continuous at a given point ξ, both $u(f,x)$ and $l(f,x)$ are continuous at ξ.†*

By means of Theorem IV we can obtain the complete solution of the saltus equation

$$s''_f(x)=0.$$

For from this equation it follows that $s'_f(x)$ is continuous, and hence, according to Theorem IV, both $u(f,x)$ and $l(f,x)$ are continuous functions. Conversely,

*The theorem is false for the *product* of two upper-semi-continuous functions.

†Although this theorem is near at hand and of sufficient interest to deserve mention among the general properties of functions, it does not seem to have been previously noted.

to 1; (2) ξ is not in such an interval. If (1) holds, we have $s'_f(x)=f(x)$ for $\alpha<x<\beta$, and the assertions in question follow from Theorem VIII. If (2) holds, there are points where $g(x)\neq 0, 1$ in every neighborhood of ξ; hence, according to (a), every neighborhood of ξ contains zeros of $f(x)$. Consequently, since in virtue of our saltus equation a neighborhood of ξ exists in which $f(x)\geqq 0$, we have $l(f,\xi)=0$, whence $s'_f(\xi)=u(f,\xi)$. Therefore $u(f,\xi) =f(\xi)$, which is equivalent with the upper-semi-continuity of f at ξ.

(c) *If* $g(x)=0$, $f(x)$ *is continuous at* ξ.

(a), (b) and (c) give us necessary relations between the character of the given function $g(x)$ and a solution of our saltus equation. Furthermore, however, these relations completely characterize the solutions. For suppose $f(x)$ satisfies these relations. In the first place, let ξ be such that $g(\xi)\neq 0, 1$. Then ξ is contained in an interval throughout which $g(x)\neq 0, 1$. According to (a), $f(x)=0$ throughout this interval, so that $s'_f(\xi)=0$, $s'_f(\xi)=g(\xi)f(\xi)$. In the second place, let $g(\xi)=1$. Then, as an easy consequence of (a) and (b), we have $l(f,\xi)=0$. This relation, taken in conjunction with the upper-semi-continuity of f, shows that $s'_f(\xi)=f(\xi)=g(\xi)f(\xi)$. In the third place, let $g(\xi)=0$. Then f is continuous according to (c), and hence $s'_f(\xi)=0=g(\xi)f(\xi)$. The saltus equation is thus satisfied at every point, and we have

THEOREM IX. *If $g(x)$ is a given continuous function, then the set of solutions of the saltus equation $s'_f(x)=g(x)f(x)$ is identical with the set of functions f such that* (a) *$f(x)=0$ where $g(x)\neq 0, 1$;* (b) *$f(x)$ is upper-semi-continuous where $g(x)=1$ and x does not lie in the interior of an interval throughout which $g(x)=1$;* (c) *$f(x)$ is upper-semi-continuous and possesses an everywhere dense set of zeros in the interior of every interval throughout which $g(x)=1$; and* (d) *$f(x)$ is continuous where $g(x)=0$.*

The following implications of Theorem IX deserve mention.

THEOREM X. *The only solution of the saltus equation $s'_f(x)=g(x)f(x)$, where $g(x)$ is a given continuous function of x taking nowhere the values* 0, 1, *is $f(x)\equiv 0$.*

THEOREM XI. *The only solution of the saltus equation $s'_f(x)=g(x)f(x)$, where $g(x)$ is a given continuous function taking at no point the value* 1, *and throughout no interval the value* 0, *is $f(x)\equiv 0$.*

THEOREM XII. *A necessary and sufficient condition that $f(x)$ be a solution of the saltus equation $s'_f(x)=g(x)f(x)$, where $g(x)$ is a continuous function taking nowhere the value* 1, *is that $f(x)$ be continuous, and* $=0$ *where $g(x)\neq 0$.*

THEOREM XIII. *If no interval exists throughout which the continuous function $g(x)$ is 0 or 1, then all the solutions of the saltus equation $s'_f(x) = g(x)f(x)$ may be obtained by making $f(x)$ an arbitrary non-negative, upper-semi-continuous function in the set of points where $g(x)=1$ and giving it the value 0 elsewhere.**

Section 3. The Saltus Equation $s''_f(x)=g(x)s'_f(x)$, where $g(x)$ is a Continuous Function.

(a) *If $g(\xi) \neq 0, 1$, then $f(x)$ is continuous at ξ.* For an interval (α, β) with ξ as interior point exists, such that $g(x) \neq 0, 1$ for $\alpha \leqq x \leqq \beta$. Since $s''_f(x)$ is 0 in an everywhere dense set, it follows from our saltus equation that $s'_f(x)=0$ is an everywhere dense subset of (α, β). We now see that $s'_f(x)$ possesses in (α, β) the properties sufficient, according to Theorem VIII, to make it a solution of the saltus equation $s'_F(x)=F(x)$. Hence $s(s'_f, x)=s'_f(x)$; *i. e.*, $s''_f(x)=s'_f(x)$ in (α, β). In particular, $s''_f(\xi)=s'_f(\xi)$, from which we would conclude, contrary to our assumption, that $g(\xi)=1$, unless $s'_f(\xi)=0$. Accordingly $f(x)$ is continuous at ξ.

(b) *If $g(\xi)=0$ and ξ is not an interior point of an interval where $g(x)$ is constantly 0, then $f(x)$ is continuous at ξ.* For since $s''_f(\xi)=0$, it follows that $s'_f(x)$ is continuous at ξ. As every neighborhood of ξ contains points where $g(x) \neq 0, 1$, we conclude from (a) that every neighborhood of ξ contains points where $s'_f(x)=0$. Hence $s'_f(\xi)=0$, and $f(x)$ is continuous at ξ.

(c) *If $g(x)=0$ throughout the interior of an interval (α, β), then $f(x)$ is continuously bounded for $\alpha < x < \beta$* (Theorem V).

(d) *If $g(\xi)=1$, then $f(x)$ is pointwise discontinuous at ξ* (Theorem VI).

Just as in the preceding section, the necessary conditions (a), (b), (c) and (d) upon a solution $f(x)$ of our present saltus equation are sufficient to characterize it completely. For suppose $f(x)$ satisfies these conditions. In the first place, let $g(\xi) \neq 0, 1$; then $f(x)$ is continuous at ξ and hence $s'_f(\xi)=0$. Since $s'_f(x)$ is upper-semi-continuous and non-negative, we have $s''_f(\xi) \leqq s'_f(\xi)$. Therefore $s''_f(\xi)=0$ and the equation $s''_f(x)=g(x)s'_f(x)$ is satisfied for $x=\xi$. In the second place, let $g(\xi)=0$ and let furthermore ξ be such that it is not an interior point of an interval where $g(x)$ is constantly 0; then $f(x)$ is continuous at ξ. Hence $s'_f(\xi)=0$, and therefore $s''_f(\xi)=0$, and $s''_f(\xi)=g(\xi)s'_f(\xi)$. That (c) and (d) are sufficient conditions follows from Theorems V and VI, respectively. We thus have

* The notion here employed of an upper-semi-continuous function defined in an arbitrary linear point-set requires no further explanation.

THEOREM XIV. *The set of solutions of the saltus equation* $s''_f(x) = g(x)s'_f(x)$, *where* $g(x)$ *is a given continuous function, is identical with the set of functions that are pointwise discontinuous at every point where* $g(x)=1$, *continuously bounded in the interior of every interval where* $g(x)$ *is constantly* 0 *and elsewhere continuous.**

A number of consequences of Theorem XIV deserve special mention:

THEOREM XV. *If the continuous function* $g(x)$ *is* 1 *nowhere and* 0 *throughout no interval, then the saltus equation* $s''_f(x)=g(x)s'_f(x)$ *has only the trivial solution* $f(x) =$ *continuous function.*

THEOREM XVI. *If the continuous function* $g(x)$ *is nowhere* $=1$, *the set of solutions of the saltus equation* $s''_f(x)=g(x)s'_f(x)$ *is identical with the set of continuously bounded functions that are continuous where* $g(x) \neq 0$.

THEOREM XVII. *If there is no interval throughout which the continuous function* $g(x)$ *is* 0 *or* 1, *then all the solutions of the saltus equations* $s''_f(x) = g(x)s'_f(x)$ *are obtained by assigning arbitrary values to* $f(x)$ *at the points where* $g(x)=1$ *and making* $f(x)$ *continuous elsewhere.*

Appendix.

A. The coefficients in the saltus equations considered in this paper have so far been exclusively continuous functions. Without attempting a comprehensive treatment for the case of discontinuous coefficients, we shall now give a theorem relating to the saltus equation $s''_f(x)=g(x)s'_f(x)$ for a discontinuous g.

THEOREM XVIII. *If the zeros of* $g(x)$ *form (at most) a non-residual set*† *in every interval, then the set of solutions of the saltus equation* $s''_f(x)=g(x)s'_f(x)$ *is identical with the set of pointwise discontinuous functions that are continuous where* $g(x) \neq 1$.

For since $s'_f(x)$ is upper-semi-continuous, it is continuous in a residual set,‡ hence $s''_f(x)=0$ in a residual set. Therefore, according to the equation $s''_f(x)=g(x)s'_f(x)$, the product $g(x)s'_f(x)$ is equal to zero in a residual set. By hypothesis, the zeros of $g(x)$ constitute a non-residual set in every interval;

*A part of the content of Theorem XIV may be derived as follows: Taking the saltus of each side of the equation $s''_f(x)=g(x)s'_f(x)$, we obtain the equation $s'''_f(x)=g(x)s''_f(x)$. But according to Sierpiński's theorem, $s'''_f(x) \equiv s''_f(x)$. Therefore either $s''_f(x)=0$ or $g(x)=1$. In similar fashion, we may deal with the saltus equation of Section 2.

† According to Denjoy, an "*exhaustible*" set (= set of first category) is the sum of a denumerable set of non-dense sets; and a "*residual*" set, the complementary set of an exhaustible set. See *Journal de Mathématiques*, Sér. 7, Vol. I (1915), pp. 122–125.

‡ See, for example, Hobson, "The Theory of Functions of a Real Variable" (1907), p. 240.

consequently, the zeros of $s_f'(x)$ constitute a non-exhaustible set in every interval. The zeros of $s_f'(x)$ are therefore everywhere dense. Since additionally $s_f'(x)$ is upper-semi-continuous, we conclude from Theorem VIII that $s_f''(x)=s(s_f', x)=s_f'(x)$, which shows according to Theorem VII that f must be pointwise discontinuous. Furthermore, we now have two possibilities: (1) Either $s_f'(x)=0$, and then f is continuous at x. Or (2) $s_f'(x)\neq 0$, and then $g(x)=s_f''(x)/s_f'(x)=1$.

We have thus shown that f must necessarily be pointwise discontinuous, and continuous where $g(x)\neq 1$. But these conditions are also sufficient. For from the pointwise discontinuity of f, it follows that $s_f''(x)=s_f'(x)$, and hence our saltus equation is satisfied where $g(x)=1$; and from the continuity of f at other points, it follows that the saltus equation is also satisfied elsewhere.

One of the consequences of Theorem XVIII is

THEOREM XIX. *The only solutions of the saltus equation $s_f''(x)=g(x)s_f'(x)$, where $g(x)$ is a given function taking the value* 1 *nowhere and the value* 0 *in a non-residual set in every interval, are continuous functions.*

B. Although we have assumed throughout the paper that $f(x)$ is single-valued and bounded at every point, it may be readily seen that the dropping of the former assumption would necessitate hardly any change in our presentation, while the dropping of the latter would require only slight modification of the results and proofs. The extension of the treatment to functions of more than one variable is not difficult, but we shall not enter upon it here. We shall also not deal with the corresponding saltus equations that arise when the f-saltus, the e-saltus, the z-saltus, etc.,* are employed instead of the ordinary saltus.

C. In conclusion we call attention to the simple saltus equation $s_f'(x)=g(x)$, where $g(x)$ is an arbitrary discontinuous function. We have not succeeded in obtaining a satisfactory characterization of its solutions.

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* *Annals of Mathematics, loc. cit.*, pp. 148 and 149.

Investigations on the Plane Quartic.

By Teresa Cohen.

§ 1. *Introduction.*

The plane quartic $(\alpha x)^4$ is taken in the form

$$ax_0^4+4a_1x_0^3x_1+4a_2x_0^3x_2+6hx_0^2x_1^2+12lx_0^2x_1x_2+6gx_0^2x_2^2+4b_0x_0x_1^3$$
$$+12mx_0x_1^2x_2+12nx_0x_1x_2^2+4c_0x_0x_2^3+bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4.$$

It will be convenient first to mention briefly certain well-known forms connected with it that will be made use of in this article.

Of these several arise from the polar forms, $(\alpha x)^3(\alpha y)$, $(\alpha x)^2(\alpha y)^2$, $(\alpha x)(\alpha y)^3$. Since of each of these there is an ∞^2, the placing of one, two, or three conditions on the curves they represent results, respectively, in a locus for the pole, in a set of points, and in an invariant condition to be satisfied by the quartic.

Upon the polar line the only condition that can be imposed is its identical vanishing. This, as is well known, means that the quartic has a double point and requires the vanishing of the discriminant, an A^{27}.*

The polar conic may be made to break up into two lines. The locus of poles of such degenerate conics is the Hessian H, an A^3x^6. To make these two lines coincide requires two additional conditions and gives for the quartic an invariant, shown by Dr. Thomsen † to be an A^{48}.

From the polar cubic more can be obtained. The locus of poles of polar cubics having a double point is the Steinerian Σ, an $A^{12}x^{12}$. The cubic has also the invariants S and T, of degrees 4 and 6, respectively, which give rise to the covariants of the quartic $S\equiv A^4x^4$ and $T\equiv A^6x^6$. From the relation connecting the discriminant of the cubic with these two invariants we have

$$\Sigma\equiv 64S^3+T^2,$$

a very useful form for calculating coefficients of the Steinerian whenever that may be necessary. To require that the polar cubic have a cusp is two conditions, giving rise to a set of twenty-four points with cuspidal polar cubics. The number of these points is determined by the fact that they are common

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* The notation $A^ix^j\xi^k$ is used to represent a comitant form of degree i in the coefficients of the quartic, j in x, and k in ξ.

† American Journal of Mathematics, Vol. XXXVIII (1916), p. 249.

points of S and T; therefore they are cusps of the Steinerian. There are also twenty-one points whose polar cubics have two double points and therefore break up into a line and a conic; these are the double points of the Steinerian. The line parts of these cubics will hereafter be referred to simply as "the twenty-one lines." Each of these lines meets the quartic so that the four tangents at the intersections are all on a point, which point is the corresponding point; *i. e.*, the pole of the cubic of which the line forms a part. If one of these lines be taken as $x_0=0$ and the corresponding point as $(1, 0, 0)$ then in $(ax)^4$

$$b_0=m=n=c_0=0.$$

The quartic also has certain contravariants obtained by imposing a condition on the four points in which a line cuts it. The locus of lines cutting the quartic in a self-apolar set is $s\equiv(s\xi)^4\equiv A^2\xi^4$. The locus of lines cutting in harmonic pairs is $t\equiv(t\xi)^6\equiv A^3\xi^6$.

§ 2. *The Undulation.*

If a line cuts the quartic in four consecutive points, the quartic is said to have an undulation. The invariant vanishing in this case is given by Salmon as an A^{60}. The undulation tangent is evidently a line of both s and t, and it is that special case of the twenty-one lines occurring when one of the lines is on its corresponding point. This corresponding point is the undulation itself, which is therefore a double point of Σ. Suppose the undulation to be at $(0, 1, 0)$ with the tangent x_0. Then

$$b=b_2=f=c_1=0.$$

The undulation is also a point of the Hessian, for its polar conic is

$$hx_0^2+2b_0x_0x_1+2mx_0x_2=x_0(hx_0+2b_0x_1+2mx_2),$$

a pair of lines. Let the double point of this conic be taken as $(0, 0, 1)$, which therefore becomes the corresponding point on the Steinerian. Then

$$m=0.$$

The polar cubic of $(0, 1, 0)$ is

$$a_1x_0^3+3hx_0^2x_1+3lx_0^2x_2+3b_0x_0x_1^2+3nx_0x_2^2.$$

This is made up of x_0 and a conic; its two double points are $(0, \sqrt{n}, \sqrt{-b_0})$ and $(0, \sqrt{n}, -\sqrt{-b_0})$; they are the two points of the Hessian corresponding to the double point of the Steinerian and are harmonic to the undulation point and to the Steinerian point corresponding to the undulation considered as a Hessian point. The terms in H not containing x_0 are

$$x_1^4x_2^2\cdot -cb_0^2+x_1^2x_2^4\cdot -2cb_0n+x_2^6\cdot -cn^2=-cx_2^2(b_0x_1^2+nx_2^2)^2.$$

Therefore x_0 is a triple tangent to the Hessian.

§ 3. *The Discriminant of the Hessian.*

The Hessian and Steinerian, as is well known, are not independent curves, but are brought into one-to-one correspondence by the relation

$$(\alpha x)(\alpha y)^2 \alpha_i = 0, \qquad i=0,1,2,$$

where x is a point of the Steinerian and y a point of the Hessian. This says that the polar cubic of x has a double point y and the polar conic of y a double point x. The joins of all such pairs x and y give rise to the Cayleyan, an $A^{12}\xi^{18}$. Since x and y can not come together unless

$$(\alpha x)^3 \alpha_i = 0, \qquad i=0,1,2,$$

which is the condition that the quartic have a double point, there is a very convenient reference scheme for handling these three curves in any other case. Let $(0,1,0)$ be a point of the Hessian, $(0,0,1)$ the corresponding point on the Steinerian, so that x_0 is a line of the Cayleyan. Then

$$\alpha_1^2 \alpha_2 \alpha_i = 0, \qquad i=0,1,2,$$

or

$$m = b_2 = f = 0.$$

Now let us see under what conditions the Hessian acquires a double point. It is known to have one when the quartic does. To discover other cases let us use the above reference scheme. Then

$$H \equiv x_1^5 x_0 \cdot 2n(bh - b_0^2) + x_1^5 x_2 \cdot 2c_1(bh - b_0^2) + \text{lower terms in } x_1.$$

Therefore $(0,1,0)$ is a double point if

$$bh - b_0^2 = 0.$$

But this says that the polar conic of $(0,1,0)$, which is $hx_0^2 + 2b_0 x_0 x_1 + bx_1^2$, shall be a line counted twice, and this means the vanishing of an A^{48}. Since this line is already on $(0,0,1)$, let it be taken as x_1, so that

$$b_0 = h = 0.$$

Then the polar cubic of any point $(k,0,1)$ on x_1, namely

$$(ak+a_2)x_0^3 + 3(a_1 k + l)x_0^2 x_1 + 3(a_2 k + g)x_0^2 x_2 + 6(lk+n)x_0 x_1 x_2$$
$$+ 3(gk + c_0)x_0 x_2^2 + 3(nk + c_1)x_1 x_2^2 + (c_0 k + c)x_2^3,$$

has a double point at $(0,1,0)$; therefore the Steinerian must contain the line x_1. Two of these polar cubics have cusps, the poles of which may be made $(1,0,0)$ and $(0,0,1)$ by requiring that

$$l = n = 0.$$

The cuspidal tangent of each cubic is on the pole of the other, and the pair of them make up the tangents to the Hessian at its double point. Then

$$S \equiv -a_1^2c_1^2x_0^2x_2^2+x_0^3x_1(-ba_1a_2c_0+ba_1g^2)+\ldots\ldots+x_1x_2^3(-ba_2c_0c_1+bc_1g^2)$$
$$+ \text{ higher powers in } x_1,$$

and

$$T\equiv 8a_1^3c_1^3x_0^3x_2^3+x_0^5x_1\cdot 4ba_1^3c_0^2+\ldots\ldots+x_1x_2^5\cdot 4ba_2^2c_1^3+ \text{ higher powers in } x_1.$$

Therefore S touches x_1 and T has x_1 as a flex line at (1, 0, 0) and (0, 0, 1). Also

$$\Sigma\equiv x_1[x_0^8x_2^3\cdot 64ba_1^6c_0^2c_1^3+\ldots\ldots+x_0^3x_2^8\cdot 64ba_1^3a_2^2c_1^6$$
$$+x_1\{x_0^{10}\cdot 16b^2a_1^6c_0^4+\ldots\ldots+x_2^{10}\cdot 16b^2a_2^4c_1^6\}+\text{ higher powers in } x_1],$$

showing that x_1 divides out only once, but that (1, 0, 0) and (0, 0, 1) are more than ordinary singularities.

The Hessian also has a double point when

$$c_1=n=0.$$

The symmetry shows that (0, 0, 1) is also a double point, as can be verified from the coefficients. Then (0, 1, 0) as a Hessian point has (0, 0, 1) as the corresponding Steinerian point, and vice versa. The polar conics of (0, 1, 0) and (0, 0, 1) are, respectively,

$$hx_0^2+2b_0x_0x_1+bx_1^2 \quad\text{and}\quad gx_0^2+2c_0x_0x_2+cx_2^2.$$

If the harmonic conjugate of x_0 as to the first pair of lines is taken as x_1 and that as to the second as x_2, then

$$b_0=c_0=0.$$

This shows that x_0 is one of the twenty-one lines. To find the degree of the invariant vanishing in this case is one of the objects of this investigation. With the above reference scheme

$$S\equiv x_1x_2^3\cdot -bcgl+x_2^2\{x_0^2(cgh^2-chl^2)+x_0x_1\cdot -bca_1g+x_1^2(-bcgh-bcl^2)\}$$
$$+\text{ lower powers in } x_2,$$

while

$$T\equiv x_1x_2^5\cdot 2bc^2l^3+x_2^4\{x_0^2\cdot -3c^2h^2l^2+2x_0x_1(-3bc^2a_2hl+6bc^2a_1l^2)$$
$$+x_1^2(b^2c^2a_2^2+4b^2cg^3+12bc^2hl^2)\}+\text{ lower powers in } x_2.$$

Therefore these two curves touch at (0, 0, 1) along the line x_1, and similarly at (0, 1, 0) along the line x_2. Then

$$\Sigma\equiv x_1^2x_2^{10}\cdot 4b^2c^4l^6+x_2^9\{x_0^2x_1\cdot -12bc^4h^2l^5+x_0x_1^2\cdot 8bc^2l^3(-3bc^2a_2hl+6bc^2a_1l^2)$$
$$+x_1^3\cdot 4bc^2l^3(b^2c^2a_2^2-12b^2cg^3+12bc^2hl^2)\}+\text{ lower powers in } x_2.$$

Therefore Σ has a singularity at (0, 0, 1), and, symmetrically, at (0, 1, 0), which is something more than merely a cusp. It may be a tac-node.

These three cases make up the totality of ways in which the Hessian may acquire a double point. Therefore its discriminant, an A^{225}, must be made up simply of the three invariants attached to them.

§ 4. *The Discriminant of* $(s\xi)^4$.

The contravariant $(s\xi)^4$ is on the line x_0 if

$$b_2 = f = c = 0.$$

The point of contact is given by

$$3c_1 m\xi_1 + (-bc_0 + b_0 c_1)\xi_2 = 0.$$

Since $(0, 0, 1)$ is taken as any one of the points in which x_0 cuts the quartic, this point of contact will be on $(\alpha x)^4$ if $m=0$ (if $c_1=0$, the quartic would have an undulation). But this says that $(0, 0, 1)$ is a point of the Steinerian. Therefore the forty-eight intersections of $(\alpha x)^4$ and $(s\xi)^4$, which considered as a point curve is of order twelve, are the same as those of $(\alpha x)^4$ and Σ. This can be substantiated by finding the point equation of $(s\xi)^4$. The line equation of $(\alpha x)^4$ is $s^3 - 27t^2$. But s formed for $(s\xi)^4$ is $-12S + A^3(\alpha x)^4$, where A^3 is the invariant A given by Salmon, and t formed for $(s\xi)^4$ is

$$T + (\alpha x)^4 \cdot (t\alpha')^3 (t\alpha'')^3 (\alpha' x)(\alpha'' x).$$

Therefore the line equation of $(s\xi)^4$ is

$$[-12S + A^3(\alpha x)^4]^3 - 27[T + (\alpha x)^4 \cdot (t\alpha')^3 (t\alpha'')^3 (\alpha' x)(\alpha'' x)]^2$$
$$= -27\Sigma + (\alpha x)^4 \cdot A^{11} x^8.$$

x_0 will become a double line of s only if

$$b_0 = c_0 = m = 0,$$

all other conditions than this placing more than one restriction on the quartic. The points in which x_0 meets the quartic are $(0, 0, 1)$, $(0, k, 1)$, $(0, \omega k, 1)$, $(0, \omega^2 k, 1)$, where $k^3 = -\dfrac{4c_1}{b}$ and $\omega^2 + \omega + 1 = 0$. The tangents to the quartic at these four points are, respectively,

$$\begin{aligned} c_1 x_1 \qquad\qquad &= 0, \\ 3nkx_0 - 3c_1 x_1 + 3c_1 k x_2 &= 0, \\ 3n\omega k x_0 - 3c_1 x_1 + 3c_1 \omega k x_2 &= 0, \\ 3n\omega^2 k x_0 - 3c_1 x_1 + 3c_1 \omega^2 k x_2 &= 0, \end{aligned}$$

and they evidently have the common point $(c_1, 0, -n)$. Let $n=0$, so that this common point becomes $(1, 0, 0)$. Since among the coefficients equated to zero are b_2, m, f, then $(0, 1, 0)$ is a point of the Hessian with $(0, 0, 1)$ as its corresponding Steinerian point. Since the tangent to the Steinerian at any

point is the polar line as to the quartic of the corresponding Hessian point, the tangent to the Steinerian at (0, 0, 1) is x_1, which is the tangent to $(ax)^4$ at the same point. Then since no distinction was made among the four points in which x_0 cuts the quartic by assigning the particular coördinates (0, 0, 1) to one of them, when one of the twenty-one lines is a line of $(s\xi)^4$, the quartic and Steinerian touch four times on this line, the four tangents going through the corresponding point. Also since the tangent to the Hessian at (0, 1, 0) is x_2, the tangents at the four Hessian points corresponding to these Steinerian points are also on that corresponding point.

x_0 will also be a line of $(s\xi)^4$ if

$$b=b_2=f=0;$$

i. e., if x_0 is a stationary line of the quartic. But this leads to nothing new. Therefore $(s\xi)^4$ has a double line only when one of the twenty-one lines is on $(s\xi)^4$, and its discriminant, an A^{54}, expresses this condition.

§ 5. *The Discriminant of* $(t\xi)^6$.

The contravariant $(t\xi)^6$ is on the line x_0 if

$$b=b_2=f=0.$$

Then

$$(t\xi)^6=2\xi_0^5\xi_1\cdot 2b_0c_1^2+\text{ lower powers in }\xi_0.$$

x_0 becomes a double line either if $b_0=0$, which says that the quartic has a double point at (0, 1, 0) with x_0 as a tangent there, or if $c_1=0$, which says that the quartic has an undulation.

$(t\xi)^6$ is also on x_0 when

$$b_2=c_1=f=0.$$

Then

$$(t\xi)^6=2\xi_0^5\xi_1\cdot -bcn+2\xi_0^5\xi_2\cdot - bcm+\text{ lower powers in }\xi_0.$$

For x_0 to be a double line requires either that $b=0$ or $c=0$, which repeats the undulation condition, or that $m=n=0$, which is the condition on the quartic under which the Hessian had two double points and for which the invariant of unknown degree vanishes. Then one of the twenty-one lines is a line of $(t\xi)^6$.

$(t\xi)^6$ also has x_0 as a line when $b=f=c=0$, but this leads to nothing new. Therefore the discriminant of $(t\xi)^6$, an A^{225}, must be made up of the discriminant of the quartic, an A^{27}, the undulation condition, an A^{60}, and the unknown invariant.

§ 6. *The Twenty-One Lines.*

The twenty-one lines* are given by an $A^i x^{21}$, where i is as yet unknown. That it can be determined is due to the fact that these lines are part of the common lines of two curves, all of whose common lines are known. These curves are the Cayleyan, an $A^{12}\xi^{18}$, and an $A^{15}\xi^{24}$, which is the locus of lines cutting the quartic so that three of the tangents at the intersections are on a point.† The common lines of these curves are are: ‡ (1) the twenty-one lines counted sixteen times, since they are quadruple lines of both curves; (2) the twenty-four stationary lines, given by an $A^{24}x^{24}$, counted twice because they are double lines of $A^{15}\xi^{24}$; (3) the forty-eight lines of $(s\xi)^4$ at its intersections with the quartic. Since the polar points of these lines are on $(\alpha x)^4$, they are lines of the curve $(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)$, an $A^9\xi^{12}$, and are obtained by the elimination of ξ from $A^9\xi^{12}$, $(s\xi)^4 \equiv A^2\xi^4$, and (ξx) as an $A^{60}x^{48}$. The eliminant of $A^{12}\xi^{18}$, $A^{15}\xi^{24}$, and (ξx) is an $A^{558}x^{432}$, which must be made up of the common lines together with such invariants as express the fact that they become indeterminate. The only such invariant is the discriminant of the quartic, for if the quartic has a double point it divides out at least once from each of these curves.

$$A^{558}x^{432} = (A^i x^{21})^{16} \cdot (A^{24}x^{24})^2 \cdot A^{60}x^{48} \cdot (A^{27})^k.$$
$$\therefore\ 558 = 16i + 108 + 27k, \quad 16i + 27k = 450.$$

This equation shows that i must be a multiple of 9, and it can be satisfied only by $i = 18$. Therefore the twenty-one lines are given by an $A^{18}x^{21}$.

§ 7. *The Invariant of Unknown Degree.*

The degree of the form giving the twenty-one lines being known, we can force one of them to lie on either $(s\xi)^4$ or $(t\xi)^6$. The condition so obtained will contain the undulation condition, however, for the tangent at an undulation is a special case of one of the twenty-one lines, and is a line of both s and t. Suppose the $A^{18}x^{21}$ factored into its component lines, the coördinates of each of

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* A curve on these lines is

$$A^9\xi^{12} \equiv 2(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha) - 3(s\xi)^4(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^2(s'''\alpha)^2 + \tfrac{1}{3}(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\alpha)^4.$$

† AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIX (1917), p. 227.

‡ Let $b_2 = m = f = 0$, so that x_0 is a line of the Cayleyan. Let $n = 0$, so that x_2 is the tangent to the Hessian at (0, 1, 0). The tangent to the Steinerian can not be chosen as the other reference line because the stationary lines are known to be among the common lines of the two curves and the Steinerian is tangent to them. Instead, let (1, 0, 0) be determined by the intersection of x_2 with the polar line of (0, 0, 1). Then $c_0 = 0$. The coefficient of ξ_0^{24} in $A^{15}\xi^{24}$ becomes $-\frac{10}{3}b^2cb_0^4c_1^8$. If $b = 0$, x_0 is a stationary line of the quartic. If $c = 0$, the quartic and Σ meet at (0, 0, 1) and x_0 is a line of $(s\xi)^4$. If $b_0 = 0$, x_0 is one of the twenty-one lines. If $c_1 = 0$, an invariant condition is imposed on the quartic.

these substituted in $(s\xi)^4$, the twenty-one resulting terms multiplied together, and the coefficients of the $A^{18}x^{21}$ substituted for symmetric functions of these coördinates. The result is an

$$A^{18\cdot 4+2\cdot 21}=A^{114}=A^{60}\cdot A^{54}.$$

This leads to nothing new. But the same process applied to $(t\xi)^6$ gives the degree of the required invariant. For we obtain $A^{18\cdot 6+21\cdot 3}=A^{171}$ from which the undulation condition must divide out at least once. Therefore the invariant in question may be either of degree 111 or 51. The first of these will not fit into the discriminant of the Hessian, but the second would give

$$A^{225}=A^{27}\cdot(A^{48})^2\cdot(A^{51})^2,$$

while for the discriminant of $(t\xi)^6$ we should have

$$A^{225}=(A^{27})^2\cdot(A^{60})^2\cdot A^{51}.$$

Therefore we have an invariant of degree 51, the vanishing of which expresses the fact that the Hessian has two double points, that $(t\xi)^6$ has a double line, and that one of the twenty-one lines is a line of $(t\xi)^6$. It also has another meaning, as will appear later.

§ 8. *Certain other Invariants.*

The degree of the form giving the twenty-one lines being known, that for the twenty-one corresponding points can be calculated from the fact that each point is the polar point of the corresponding line as to both $(s\xi)^4$ and $(t\xi)^6$. Suppose, as before, the $A^{18}x^{21}$ factored into its component lines, the coördinates substituted for η in $(s\xi)(s\eta)^3$, and the symmetric functions of these coördinates in the product of the resulting terms replaced by the coefficients of $A^{18}x^{21}$. The result gives the twenty-one points together with such invariants as express that the polar point becomes indeterminate (*i. e.*, that one of the lines be a double line of s). Then

$$A^{18\cdot 3+2\cdot 21}\xi^{21}=A^{54}\cdot A^{42}\xi^{21}.$$

Similarly from $(t\xi)^6$ we obtain

$$A^{18\cdot 5+3\cdot 21}\xi^{21}=A^{60}\cdot A^{51}\cdot A^{42}\xi^{21}.$$

Therefore the twenty-one points are given by an $A^{42}\xi^{21}$.* Suppose one of th— to be on $(\alpha x)^4$, and therefore on its own polar cubic. Then, as before, considering $A^{42}\xi^{21}$ broken up into its component points, the coördinates substituted

* This degree fits in with the form $A^{36}\xi^{18}$ for the line equation of Σ. For the cusps are obtained from the eliminant of S, T, and (ξx) as an $A^{48}\xi^{24}$. Then from the Plücker formula

$$A^{12\cdot 22}\xi^{12\cdot 11}=A^{264}A^{132}=(A^{42}\xi^{21})^2\cdot(A^{48}\xi^{24})^3\cdot A^{36}\xi^{18}.$$

in $(\alpha x)^4$, and their symmetric functions replaced in the product of the resulting terms, we obtain an $A^{42\cdot 4+21}=A^{189}$. Out of this the condition for an undulation, when the point is on the line part of its polar cubic, must divide out at least once. If it is no more than once, we have left an A^{129} to express that the point is on the conic part of its polar cubic. To substantiate this degree we must find some case into which the undulation condition does not enter. We had before that the point is $(1, 0, 0)$ and its corresponding line x_0 if $b_0=m=n=c_0=0$. To put the point on the quartic requires that $a=0$. The undulation case is excluded because our choice of coördinates for the point and line implies that they are not incident. We can still choose $(0, 1, 0)$ and $(0, 0, 1)$ on x_0. Let us take them as the double points of the polar cubic of $(1, 0, 0)$, so that $g=h=0$. Then in $(s\xi)^4$ the coefficients of ξ_1^4, ξ_2^4, $\xi_1^3\xi_2$, $\xi_1\xi_2^3$ vanish, showing that x_1 and x_2 are lines of s with contact $(1, 0, 0)$, so that $(1, 0, 0)$ is a double point of $(s\xi)^4$.* It is known that the double lines of $(\alpha x)^4$ are given by an $A^{16}x^{28}$; therefore the double points of $(s\xi)^4$ are given by an $A^{32}\xi^{28}$. One of them can lie on $(\alpha x)^4$ only in our present case or when $(\alpha x)^4$ has a double point, for them $(s\xi)^4$ has that same double point with the same tangents. Then by the method of considering the $A^{32}\xi^{28}$ factored into its component parts we have as the condition that one of these points be on $(\alpha x)^4$ an

$$A^{32\cdot 4+28}=A^{156}=A^{129}\cdot A^{27}.$$

This establishes the degree of the A^{129}.

The totality of polar cubics of the twenty-one points given by the $A^{42}\xi^{21}$ is given by an $A^{63}x^{63}$. Suppose this factored into its components cubics $(\beta_i x)^3$, and then each of these factored into its component line and conic $(\gamma_i x)\cdot(\delta_i x)^2$. Then the condition that the line touch the conic is of second degree in both γ_i and δ_i, therefore of second degree in β_i. Therefore the invariant expressing the condition that there be a polar cubic made up of a conic and a line touching it is an $A^{63\cdot 2}=A^{126}$. This result can be substantiated in other ways. If, as usual, we take the line as x_0 and the corresponding point as $(1, 0, 0)$ so that

$$b_0=m=n=c_0=0,$$

then the conic will touch x_0 at $(0, 1, 0)$ if

$$h=l=0.$$

Then S goes through $(1, 0, 0)$ while T has a double point there, so that Σ has two cusps and a double point coming together and acquires a triple point, since x_0^9 is the highest power of x_0 appearing in its equation. Since two of the

*An examination of the coefficients of $(s\xi)^4$ shows that an undulation is not a double point of it.

intersections of S and T have come together, the A^{126} must be a factor of the tact-invariant of these two curves. We have seen (§ 3) that in case of the vanishing of either the A^{48} or the A^{51} that S and T touch at two distinct points, and these three cases seem to be the only ones in which two intersections of these curves come together. Their tact-invariant is found from the formula given by Salmon to be of degree $4\cdot 6(6+8-3)+6\cdot 4(4+12-3)=576$. Then

$$A^{576}=(A^{48})^2\cdot(A^{51})^2\cdot(A^{126})^3.$$

Also, inspection of their coefficients shows that both H and S touch x_0 at $(0, 1, 0)$. As will be seen later, two intersections of S and H can come together only when two of S and T do. When the A^{51} vanishes H has two double points on S; when the A^{48} vanishes, H has a double point which is also a double point of S. The tact-invariant of S and H is of degree $4\cdot 6(6+8-3)+3\cdot 4(4+12-3)=420$.

$$A^{420}=(A^{48})^4\cdot(A^{51})^2\cdot A^{126}.$$

The twenty-one polar cubics being given by an $A^{63}x^{63}$, while the twenty-one line parts are given by an $A^{18}x^{21}$, the twenty-one conic parts are given by an $A^{45}x^{42}$. Since the discriminant of a conic is of third degree in its coefficients, the condition that one of these conics break up is an A^{135}. But then the polar cubic consists of three lines, any two of which may be considered as the conic part, so that this degree is probably divided by three. Therefore the condition that there be a polar cubic made up of three lines is an A^{45}. Then three of the double points of the Steinerian have come together, giving it a triple point.*

§ 9. *The Eliminant of S, T, H.*

Under the reference scheme where

$$b_3=f=m=0,$$

the coefficient of the highest term in x_2 in both S and T is a power of (c_1l+n^2). Then both coefficients vanish if $c_1=n=0$, but this, as we have seen, means the

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* Professor Morley has pointed out to me that the degrees of these last two invariants can be substantiated by the work of Caporali (Memorie di Geometria, p. 171) on a web of plane curves. If i curves of the web are of a particular kind, then the invariant expressing that there be a curve of that kind in a net is of degree i in the coefficients of the three curves upon which the net is built up. From his results the invariants expressing that in a net of cubics there be a cubic with three double points or a cubic with two coincident double points are of degree 15 and 42, respectively, in the coördinates of the three cubics. But in the polar net each cubic is linear in the coefficients of the quartic. Therefore the invariants are are of degree 45 and 126 in the coefficients of the quartic.

vanishing of an A^{51}. Therefore for the general quartic we obtain (0, 0, 1) as one of the common points of S and T if we make

$$l=n=0.$$

But then the coefficient of x_1^4 vanishes in S. Therefore to the twenty-four cusps of the Steinerian correspond those twenty-four points of the Hessian which make up its intersection with S.

It is known that the polar cubic of a point on S can be reduced to the sum of the cubes of three linear factors; its Hessian is the product of these factors and is one of those four triangles whose sides pass through all the flexes of the cubic. If the cubic has a double point, there is only one such proper triangle, made up of the tangents at the double point and a line through the three remaining flexes (flex line); in case of a cusp even this degenerates to the cusp tangent counted twice and the line joining the cusp to the sole remaining flex. Here the polar cnbic of (0, 0, 1) is

$$a_2x_0^3+3gx_0^2x_2+3c_0x_0x_2^2+3c_1x_1x_2^2+cx_2^3,$$

and its Hessian is

$$-c_1^2x_2^2(a_2x_0+gx_2).$$

But x_2 is the tangent to H at (0, 1, 0) and $a_2x_0+gx_2$ is the tangent to S at the same point. Therefore the polar cubic of a cusp of the Steinerian has as its cusp tangent and its flex line the tangents to H and S, respectively, at the corresponding point. The cubic can be thrown into the form

$$\frac{1}{a_2^2}\left[(a_2x_0+gx_2)^3+x_2^2\{3a_2(a_2c_0-g^2)x_0+3a_2^2c_1x_1+(ca_2^2-g^3)x_2\}\right]$$

which also shows up the tangent at the flex as the linear form in brackets.

The polar cubic of (0, 1, 0) is

$$a_1x_0^3+3hx_0^2x_1+3b_0x_0x_1^2+bx_1^3+c_1x_2^3$$

and its Hessian is

$$c_1x_2[x_0^2(a_1b_0-h^2)+x_0x_1(ba_1-b_0h)+x_1^2(bh-b_0^2)]$$

Then the tangent to H at (0, 1, 0) passes through three flexes of the cubic, the three flex tangents pass through the corresponding Steinerian point and the binary Hessian of these three tangents picks up the six remaining flexes.

The coefficient of x_2^6 in H is $-c_1^2g$. Since we cannot have $c_1=0$ without bringing on the vanishing of the A^{51} and worse, let us make $g=0$. Then we have made S, T, H have a common point at (0, 0, 1). The polar cubic of (0, 0, 1) becomes

$$a_2x_0^3+x_2^2(3c_0x_0+3c_1x_1+cx_2).$$

If we let the intersection of stationary tangent and cusp tangent be $(1, 0, 0)$, then

$$c_0 = 0.$$

But since we now have $g = n = c_0$, the Steinerian point corresponding to $(0, 0, 1)$ as a Hessian point is $(1, 0, 0)$, which also becomes a cusp on the Steinerian. So to sum up, we have: (1) the cusp e_0 on the Steinerian corresponds to e_2 on the Hessian; (2) the cusp e_2 on the Steinerian corresponds to e_1 on the Hessian; (3) the tangent to the Hessian at e_1 goes through e_0 and at e_2 goes through e_1; (4) the polar cubic of e_2 has the tangent to the Hessian at e_1 as cusp tangent, the line $x_0 (= e_1 e_2)$ as flex line, and a stationary tangent going through e_0; (5) the polar cubic of e_0 has an analogous set of lines; (6) S is tangent to x_0 at e_1 and to x_1 at e_2.

The real underlying significance of this state of affairs is not known; neither is the degree of the invariant representing it, but it can be guessed at. For the eliminant of S, T, H is an A^{360}. The only other condition under which these three curves have a common point is when the A^{51} vanishes, but then they have four common points, for H has two double points on the other curves. Then presumably the A^{51} divides four times out of the eliminant, leaving an A^{156}. The essential difference between the two cases seems to be this: To every ST point (intersection of S and T) there is attached an SH point with which it cannot coincide without more than one condition on the quartic. In order for S, H, T to have a common point an ST point must coincide with an SH point. If then the corresponding SH and ST points also coincide, the A^{51} vanishes. When there is no further coincidence, we have the other case.

§ 10. *The Eliminant of the Polar Cubic, Conic, and Line.*

Suppose we ask that the polar cubic, conic, and line of $(0, 1, 0)$, which is not on the quartic, have a common point $(0, 0, 1)$. Then

$$b_2 = f = c_1 = 0.$$

The symmetry of this condition shows that the relation between $(0, 1, 0)$ and $(0, 0, 1)$ is a mutual one. Also x_0 is a line of $(t\xi)^6$. Therefore we might define $(t\xi)^6$ as the locus of the joins of pairs of points so related that the polar curves of each pass through the other, the pair of points being the common harmonic pair of the two harmonic pairs of points in which their join cuts the quartic.

In general the eliminant of the polar curves of x, which are an Ax^3y, an Ax^2y^2, and an Axy^3, is an $A^{11}x^{26}$. Out of this $(\alpha x)^4$ divides twice, leaving an

A^9x^{18}. Before asking the meaning of this latter curve, let us examine the eliminant of a cubic Ay^3, its polar conic Axy^2, and its polar line Ax^2y. It is an $A^{11}x^{15}$, from which the cubic divides out twice. The remainder, an A^9x^9, gives the nine stationary lines. Therefore the A^9x^{18} is for the quartic the locus of points lying on the stationary lines of their polar cubics, also the locus of flexes of polar cubics the tangent at which lies on the pole.

Salmon gives as the equation of the nine stationary lines of the cubic.

$$5SU^2H-H^3-U\theta,$$

where S is the invariant of degree 4, U is the cubic itself, H is its Hessian and θ is a certain A^8x^6. For a polar cubic of the quartic the first three pass over readily enough into the covariant S, the quartic itself, and its Hessian, but θ is not so easily transferred. So we will substitute for it another A^8x^6, expressible in terms of S, U, H, and θ, which gives the locus of a point whose polar conic as to the cubic is on its polar line as to the Hessian. Then it is easily shown that

$$A^9x^{18}=8SU^2H-H^3+9U\theta',$$

where S is the covariant S of the quartic, U the quartic itself, H its Hessian, and θ' the locus of those points whose polar conics as to the quartic are on their polar lines as to the Hessian.

The form of the A^9x^{18} shows that all of its intersections with $(\alpha x)^4$ are used up at the flexes of the quartic. The twenty-four points are also part of its intersection with H, but there are eighty-four more, lying on θ'. Now we saw that $(0, 1, 0)$ is a point of the A^9x^{18} if

$$b_2=f=c_1=0.$$

To make it a point of H also requires that

$$m=0.^*$$

But there are only forty-two such points, for they are the points where H, $(t\xi)^6$ and the Cayleyan all touch.† Then the intersections of H and θ' must come together by twos, and, indeed, it is readily shown that the two curves touch at their common points.

It is also easily shown that the forty-two inflexions of the Steinerian are at the points corresponding to these Hessian points.

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*Or symmetrically, $(0, 0, 1)$ is a point of H if $n=0$.

†Proc. Nat. Ac. Sci., Vol. III (1917), p. 449.

§ 11. *The Polar Conic of Two Points.*

The eliminant of the polar curves of the quartic connects up with the polar conic of two points in the following way. The polar conic of x and y as to the quartic is $(\alpha x)(\alpha y)(\alpha z)^2$. Any point on the line xy may be represented parametrically as $x+\lambda y$. This line meets the conic in two points whose parameters λ are given by

$$(\alpha x)(\alpha y)(\overline{\alpha x+\lambda y})^2=(\alpha x)^3(\alpha y)+2\lambda(\alpha x)^2(\alpha y)^2+\lambda(\alpha x)(\alpha y)^3=0.$$

If we ask that these points become indeterminate, we ask that the coefficients of λ all vanish. Then x and y are a pair of points such that the polar curves of each are on the other, so that we can define the A^9x^{18} of the last section as the locus of pairs of points whose polar conic is not only degenerate but contains their join, and $(t\xi)^6$ as the locus of this join. If x and y be taken as $(0, 1, 0)$ and $(0, 0, 1)$, then, as we have seen,

$$b_2=f=c_1=0.$$

The polar conic of the two points is

$$lx_0^2+2mx_0x_1+2nx_0x_2=x_0(lx_0+2mx_1+2nx_2).$$

To make the double point at one of the points requires that

$$m=0 \quad (\text{or } n=0).$$

But this makes $(0,1,0)$ one of the forty-two points on H where it touches $(t\xi)^6$.

In general for a given y we have a whole locus of points x, the polo-Hessian of y to be exact, such that the polar conic of x and y is a pair of lines. We have seen that for one of these lines to be xy, x and y must lie on an A^9x^{18}. If we ask instead that the double point be at x, then x is a point of the Hessian and y a point of the Steinerian. For the polar conic of $(0,1,0)$ and $(0,0,1)$ is

$$lx_0^2+2mx_0x_1+2nx_0x_2+b_2x_1^2+2fx_1x_2+c_1x_2^2,$$

and this becomes two lines on $(0, 1, 0)$ if

$$m=b_2=f=0,$$

being, in fact, the tangents at the double point of the polar cubic of the Steinerian point. To ask both that xy be one of the lines and that the double point be at x picks out, as we have seen, the forty-two inflexions of the Steinerian, and the forty-two points where the Hessian touches the Cayleyan and $(t\xi)^6$.

Making a fresh start, let us ask that the polar conic of $(0,1,0)$ and $(0,0,1)$ be the square of the line. If we take $(1, 0, 0)$ as a point of this line, then

$$l=m=n=0, \quad b_2c_1-f^2=0.$$

But then we find that the coefficients of x_1^4 and x_2^4 in S vanish. Therefore the pair of points whose polar conic is a repeated line lie on S. But we already have a correspondence of the points of S, for the polo-Hessian of a point of S is made up of three lines, the intersections of which are on S. With the above reference scheme

$$S_{400}=(a_1b_0-h^2)(-a_2c_0+g^2).$$

(1, 0, 0) was any point of the repeated line. Let us take it as one of the intersections with S by making

$$a_1b_0-h^2=0.$$

Then the polo-Hessian of (0, 1, 0) becomes

$$x_1[x_0x_1(ba_1f-a_1b_2^2-b_0fh)+x_0x_2(ba_1c_1-a_1b_2f-b_0c_1h)$$
$$+x_1^2(bfh-b_0^2f-b_2^2h)+x_1x_2(bc_1h-b_0^2c_1-b_2fh)].$$

The double points are (1, 0, 0), (0, 0, 1), and

$$y\equiv(bc_1h-b_0^2-b_2fh,\ -ba_1c_1+a_1b_2f+b_0c_1h,\ ba_1f-a_1b_2^2-b_0fh).$$

Then the polar conic of (0, 1, 0) and (1, 0, 0) becomes $a_1x_0^2+2hx_0x_1+b_0x_1^2$, that of (0, 1, 0) and (0, 0, 1) is $b_2x_1^2+2fx_1x_2+c_1x_2^2$, and that of (0, 1, 0) and y is x_1^2. Therefore the polar conic of a point on S and any vertex of its polo-Hessian triangle is the square of the opposite side.*

If, now, we also ask that the repeated line be on (0, 1, 0) we have

$$b_2=f=0.$$

Then (0, 1, 0) is one of the twenty-four intersections of S and H. Therefore the polar conic of an SH point and its corresponding ST point is the square of a line on the SH point (the tangent to H there).

If, finally, we ask that the polar conic of (0, 1, 0) and (0, 0, 1) be x_0^2, we have

$$m=n=b_2=f=c_1=0.$$

But then the A^{51} vanishes. Therefore we have a new definition for the A^{51}; it is the invariant expressing the condition that there be a pair of points such that their polar conic as to the quartic is the square of the line joining them.

§ 12. *Salmon's Connex.*

Salmon has shown that with any plane curve $(\alpha x)^n$ there is associated an $A^8x^{2(n-2)}y^{n-2}$, which when x is a point on the curve picks out the remaining

* We might expect this sort of (3, 1) correspondence from the following considerations: The line equation of $(\alpha x)(\alpha y)(\alpha z)^2$ is an $A^2x^2y^2\xi^2$. If the conic is the square of a line, the line equation vanishes identically. Equating each coefficient to zero, we have six equations from which we can eliminate the six quantities y_0^2, y_0y_1, y_0y_2, y_1^2, y_1y_2, y_2^2, giving an $A^{12}x^{12}$. This must be S^3.

intersections of the tangent at x with the curve. This connex is expressible in terms of polars of the Hessian of the curve itself and the Hessians of the polar curves of x as to $(ay)^*$. In case of the quartic the connex is an $A^3x^4y^2 \equiv 15\,(hx)^4(hy)^2 - 9\,(h_1x)\,(h_1y)^2$, where $(hy)^6$ is the Hessian of the quartic and $(h_1y)^3$ is the Hessian of the polar cubic of x. Explicitly

$$
\begin{aligned}
A^3x^4y^2 = \overset{3}{\Sigma} x_0^4 [& y_0^2 \cdot 6\,(agh - al^2 - a_1^2g - a_2^2h + 2a_1a_2l) \\
& + \overset{2}{\Sigma} y_0y_1 \cdot 4\,(ab_0g + ahn - 2alm - a_2^2b_0 - a_1^2n + 2a_1a_2m - a_1gh + a_1l^2) \\
& + \overset{2}{\Sigma} y_1^2 (abg - ba_2^2 + ab_0n - 2ab_2l + afh - am^2 + 2a_1a_2b_2 - a_1^2f - a_1b_0g \\
& \qquad - a_1hn + 2a_1lm) \\
& + y_1y_2 (ab_0c_0 + 2ab_2g + 2ac_1h - 4afl - amn - 2a_1^2c_1 - 2a_2^2b_2 \\
& \qquad - 4a_1a_2f - a_1c_0h - a_2b_0g - a_1gm - a_2hn + 2a_1ln + 2a_2lm)] \\
+ \overset{6}{\Sigma} x_0^3x_1 [& y_0^2 \cdot 8\,(ab_0g + ahn - 2alm - a_2^2b_0 - a_1^2n + 2a_1a_2m - a_1gh + a_1l^2) \\
& + y_0y_1 \cdot 2\,(abg - ba_2^2 + 7ab_0n - 2ab_2l + afh - 7am^2 + 2a_1a_2b_2 - a_1^2f \\
& \qquad + 5a_1b_0g - 12a_2b_0l - 7a_1hn + 12a_2hm + 2a_1lm - 6gh^2 + 6hl^2) \\
& + y_0y_2 \cdot (7ab_0c_0 + 2ab_2g + 2ac_1h - 4afl - 7amn - 2a_1^2c_1 - 2a_2^2b_2 \\
& \qquad + 4a_1a_2f - 7a_1c_0h - 7a_2b_0g + 17a_1gm + 17a_2hn - 10a_1ln \\
& \qquad - 10a_2lm - 12ghl + 12l^3) \\
& + y_1^2 \cdot 3\,(abn + ab_0f + ba_1g - 2ab_2m - 2ba_2l + 2a_2b_2h - a_1fh - b_0gh \\
& \qquad - hn^2 + 2hlm) \\
& + y_1y_2 (abc_0 + 5ab_0c_1 - ba_2g + 3ab_2n - 9afm + 2a_1b_0c_0 + 7a_1b_2g \\
& \qquad - 5a_1c_1h - 4a_2b_0n - 10a_2b_2l + 11a_2fh - 3c_0h^2 - 2a_1fl - 3b_0gl \\
& \qquad - 2a_1mn + 4a_2m^2 - 3ghm + 3hln + 6l^2m) \\
& + y_2^2 (2acb_0 + ab_2c_0 - 2ca_1h - 3ac_1m - 2a_2b_0c_0 - a_2b_2g + 5a_2c_1h \\
& \qquad + 4a_1c_0m - 2a_1c_1l + 4a_1fg - 4a_2fl - 3c_0hl - 4a_1n^2 + 2a_2mn \\
& \qquad - 3glm + 6l^2n)] \\
+ \overset{3}{\Sigma} x_0^2x_1^2 [& \overset{2}{\Sigma} y_0^2 \cdot 3\,(abg - ba_2^2 + 3ab_0n - 2ab_2l + afh - 3am^2 + 2a_1a_2b_2 - a_1^2f \\
& \qquad + a_1b_0g - 4a_2b_0l - 3a_1hn + 4a_2hm + 2a_1lm - 2gh^2 + 2hl^2) \\
& + y_0y_1 \cdot 6\,(abn + ab_0f + ba_1g - 2ab_2m - 2ba_2l + 2a_2b_2h + 4a_1b_0n \\
& \qquad - a_1fh - b_0gh - 4a_1m^2 - 4b_0l^2 - 5h^2n + 10hlm) \\
& + \overset{2}{\Sigma} y_0y_2 \cdot 3\,(abc_0 + ab_0c_1 - ba_2g - ab_2n - afm + 2a_1b_0c_0 + 3a_1b_2g \\
& \qquad - a_1c_1h + 4a_2b_0n - 2a_2b_2l + 3a_2fh - 3c_0h^2 - 2a_1fl - 7b_0gl \\
& \qquad - 2a_1mn - 4a_2m^2 + 5ghm - hln + 6l^2m) \\
& + y_2^2 (abc - ab_2c_1 - ba_2c_0 + 2ca_1b_0 - 3ch^2 + 4a_1b_2c_0 + 4a_2b_0c_1 - 6a_1c_1m \\
& \qquad - 6b_0c_0l - 3a_2fm - 3b_2gl + 3c_0hm + 3c_1hl + 6fgh - 3fl^2 - 3gm^2 \\
& \qquad - 6hn^2 + 12lmn)]
\end{aligned}
$$

$$+\sum^{3} x_0^2 x_1 x_2 [y_0^2 \cdot 3\,(3ab_0c_0+2ab_2g+2ac_1h-4afl-3amn-2a_1^2c_1-2a_2^2b_2$$
$$+4a_1a_2f-3a_1c_0h-3a_2b_0g+5a_1gm+5a_2hn-2a_1ln-2a_2lm$$
$$-4ghl+4l^3)$$
$$+\sum^{2} y_0y_1 \cdot 3\,(abc_0+3ab_0c_1-ba_2g+ab_2n-5afm+6a_1b_0c_0+5a_1b_2g$$
$$-3a_1c_1h+4a_2b_0n-6a_2b_2l+7a_2fh-7c_0h^2-2a_1fl-13b_0gl$$
$$-6a_1mn-4a_2m^2+9ghm+hln+10l^2m)$$
$$+\sum^{2} y_1^2 \cdot 3\,(abc_1+2ba_1c_0-ab_2f-3bgl+4a_2b_0f-2b_0c_0h-4a_2b_2m$$
$$+5b_2gh-c_1h^2-2a_1fm-2b_0gm-fhl+4lm^2)$$
$$+y_1y_2\,(abc+8ab_2c_1+2ba_2c_0+2ca_1b_0-9af^2-3bg^2-3ch^2+16a_1b_2c_0$$
$$+16a_2b_0c_1-12a_1c_1m-12a_2b_2n-6a_1fn-6a_2fm-18b_0gn$$
$$-12b_2gl-18c_0hm-12c_1hl+30fgh+6gm^2+6hn^2+24lmn)].$$

This connex may be considered either as a conic or a quartic, depending on whether x or y is the given point. If the curve has a double point the conic vanishes, but the quartic does not. Let

$$a=a_1=a_2=0,$$

so that $(1, 0, 0)$ is a double point, and let this be taken as the point y. Then

$$A^3x^4y^2=\sum^{2} x_0^2x_1^2 \cdot 3\,(-2gh^2+2hl^2)+x_0^2x_1x_2 \cdot 3\,(-4ghl+4l^3)$$
$$+\sum^{2} x_0x_1^3 \cdot 3\,(-b_0gh-h^2n+2hlm)$$
$$+\sum^{2} x_0x_1^2x_2 \cdot 3\,(-c_0h^2-2b_0gl-ghm+4l^2m)$$
$$+\sum^{2} x_1^4\,(bgh-bl^2-b_0^2g-b_0hn+2b_0lm)$$
$$+\sum^{2} x_1^3x_2\,(-b_0c_0h+4b_2gh-4b_0gm+2b_0ln-4b_2l^2-3hmn+6lm^2)$$
$$+x_1^2x_2^2\,(-3b_0gn-3c_0hm+6fgh-6fl^2-3gm^2-3hn^2+12lmn).$$

Obviously this curve has a double point at $(1, 0, 0)$ with the same tangents as the quartic. Therefore six of the sixteen intersections of the two curves are used up here. To find the others let us take $(0, 1, 0)$ on $(ax)^4$, so that $b=0$. Then in order that $(0, 1, 0)$ may also be on $A^3x^4y^2$ we must have

$$b_0\,(b_0g+hn-2lm)=0.$$

If $b_0=0$, the intersection is the contact of a tangent from the double point; there are six such. If $b_0\neq 0$, let us take another intersection which is not one of these contacts as $(0, 0, 1)$. Then

$$c=0, \quad b_0g+hn-2lm=0, \quad c_0h+gm-2ln=0.$$

These last two equations say that the tangents at the double point, given by $hx_1^2+2lx_1x_2+gx_2^2=0$, are apolar to the three points in which x_0 cuts the polar cubic of the double point, these three points being given by

$$b_0\xi_2^3-3m\xi_1\xi_2^2+3n\xi_1^2\xi_2-c_0\xi_1^3=0.$$

But this is true when x_0 is the flex line of the polar cubic. This result can be verified by starting out merely with $a=a_1=a_2=0$ and throwing the polar cubic of $(1, 0, 0)$,

$$3hx_0x_1^2+6lx_0x_1x_2+3gx_0x_2^2+b_0x_1^3+3mx_1^2x_2+3nx_1x_2^2+c_0x_2^3,$$

into the canonical form by

$$h=g=m=n=0.$$

Then x_0 is the flex line. It meets $A^3x^4y^2$ in points given by

$$-l^2(bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4)=0,$$

evidently the same points as those in which x_0 meets the quartic. Therefore the four remaining intersections of $A^3x^4y^2$ and $(\alpha x)^4$ are on the flex line of the polar cubic of the double point.

This result can be verified algebraically. For

$$\begin{aligned} A^3x^4y^2=&(gh-l^2)(6hx_0^2x_1^2+12lx_0^2x_1x_2+6gx_0^2x_2^2+4b_0x_0x_1^3+12mx_0x_1^2x_2\\ &+12nx_0x_1x_2^2+4c_0x_0x_2^3+bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4)\\ &-[3hx_0x_1^2+6lx_0x_1x_2+3gx_0x_2^2+b_0x_1^3+3mx_1^2x_2+3nx_1x_2^2+c_0x_2^3]\\ &\quad[4(gh-l^2)x_0+(b_0g+hn-2lm)x_1+(c_0h+gm-2ln)x_2]\\ =&(gh-l^2)(\alpha x)^4-(\alpha x)^3(\alpha y)\cdot\text{flex line.}\end{aligned}$$

The polar cubic picks up the six intersections at the double point and those at the six contacts of tangents from the double point; the flex line gives the remainder.

The flex line can be obtained by a sort of limit process from the polar lines of $(\alpha x)^4$ and $(hx)^6$. Instead of at once taking $(1, 0, 0)$ as a double point, let us make x_0 the polar line of $(1, 0, 0)$ so that $a_1=a_2=0$. Then if y is $(1, 0, 0)$,

$$\begin{aligned} &3(\alpha y)^4\cdot(hx)(hy)^5+(hy)^6\cdot(\alpha x)(\alpha y)^3\\ &\qquad=a[4(gh-l^2)x_0+(b_0g+hn-2lm)x_1+(c_0h+gm-2ln)x_2].\end{aligned}$$

If we divide this by $(\alpha y)^4=a$ and then let $a=0$ so as to get a double point, we have the flex line.

$A^3x^4y^2$ is not the only curve which picks up the six points of contact from y when y is a double point. So does $(hx)^5(hy)$, for when the double point is taken as $(1, 0, 0)$,

$$\begin{aligned}(hx)^5(hy) = &\sum^2 x_0^3x_1^2\cdot 2(-gh^2+hl^2)+x_0^3x_1x_2\cdot 4(-ghl+l^3)\\ &+\sum^2 x_0^2x_1^3\cdot(-b_0gh-2b_0l^2-3h^2n+6hlm)\\ &+\sum^2 x_0^2x_1^2x_2(-3c_0h^2-6b_0gl+3ghm+6l^2m)\\ &+\sum^2 x_0x_1^4\cdot\tfrac{1}{3}(bgh-4bl^2-b_0^2g-4b_0hn+6b_2hl+2b_0lm-3fh^2+3hm^2)\\ &+\sum^2 x_0x_1^3x_2\cdot\tfrac{2}{3}(-3bgl-2b_0c_0h+5b_2gh-3c_1h^2-2b_0gm-2b_0ln\\ &\quad -2b_2l^2+3fhl+6lm^2)\\ &+x_0x_1^2x_2^2\cdot\tfrac{1}{3}(-3bg^2-3ch^2-6b_0c_0l-6b_0gn-6b_2gl-6c_0hm\\ &\quad -6c_1hl+18fgh+18lmn)\\ &+\sum^2 x_1^5\cdot\tfrac{1}{3}(bhn-2blm-b_0^2n+2b_0b_2l-b_0fh+b_0m^2)\\ &+\sum^2 x_1^4x_2\cdot\tfrac{1}{3}(bc_0h-2bgm-2bln-b_0^2c_0+2b_0b_2g-2b_0c_1h+4b_0fl\\ &\quad +4b_2hn-2b_0mn-2b_2lm-3fhm+3m^3)\\ &+\sum^2 x_1^3x_2^2\cdot\tfrac{1}{3}(-cb_0h-3bgn+4b_2c_0h-4b_0c_0m+2b_0c_1l+5b_0fg\\ &\quad -2b_2gm-6c_1hm-2b_0n^2-2b_2ln+3fhn+6m^2n),\end{aligned}$$

and the coefficient of x_1^5 vanishes for $b=b_0=0$. That it passes through the intersections of $(\alpha x)^4$ and $(\alpha x)^3(\alpha y)$ is shown by

$$\begin{aligned}3(hx)^5(hy) = &[6hx_0^2x_1^2+12lx_0^2x_1x_2+6gx_0^2x_2^2+4b_0x_0x_1^3+12mx_0x_1^2x_2+12nx_0x_1x_2^2\\ &+4c_0x_0x_2^3+bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4]\\ &\quad[4(gh-l^2)x_0+(b_0g+hn-2lm)x_1+(c_0h+gm-2ln)x_2]\\ &+[3hx_0x_1^2+6lx_0x_1x_2+3gx_0x_2^2+b_0x_1^3+3mx_1^2x_2+3nx_1x_2^2+c_0x_2^3]\\ &\quad[x_0^2\cdot -10(gh-l^2)+\sum^2 x_0x_1(-5b_0g-5hn+10lm)\\ &\qquad+\sum^2 x_1^2(-bg-b_0n+2b_2l-fh+m^2)\\ &\qquad\quad+x_1x_2(-b_0c_0-2b_2g-2c_1h+4fl+mn)]\\ &=(\alpha x)^4\cdot\text{flex line}-(\alpha x)^3(\alpha y)\left[\frac{1}{a}A^3x^2y^4+x_0\cdot\text{flex line}\right],\end{aligned}$$

where by $\frac{1}{a}A^3x^2y^4$ is meant the result of forming $A^3x^2y^4$ for $y\equiv(1, 0, 0)$ when only $a_1=a_2=0$, dividing by a, and then letting $a=0$. Any further pursuit of this relation, however, when the quartic has no double point, seems merely to lead to Cayley's indentity

$$3(\alpha x)^4\cdot(hx)(hy)^5-(\alpha x)^3(\alpha y)\cdot A^3x^2y^4+(\alpha x)(\alpha y)^3\cdot A^3x^4y^2 -3(\alpha y)^4\cdot(hx)^5(hy)\equiv 0.$$

Note 1.

The developments of H and S are given by Salmon only for a special quartic. Those for the general quartic may be calculated by means of a differential operator from the leading coefficients, which in case of S is obtained directly from the invariant S of the cubic as given in the *Higher Plane Curves.* The general expression for the Hessian is

$$
\begin{aligned}
&\overset{3}{\Sigma} x_0^6 (agh - al^2 - a_1^2 g - a_2^2 h + 2a_1a_2l) \\
&+\overset{6}{\Sigma} x_0^5 x_1 \cdot 2(ab_0g + ahn - 2alm - a_2^2 b_0 - a_1^2 n + 2a_1a_2m - a_1gh + a_1l^2) \\
&+\overset{6}{\Sigma} x_0^4 x_1^2 (abg - ba_2^2 + 4ab_0n - 2ab_2l + afh - 4am^2 + 2a_1a_2b_2 - a_1^2 f + 2a_1b_0g - 6a_2b_0l \\
&\qquad - 4a_1hn + 6a_2hm + 2a_1lm - 3gh^2 + 3hl^2) \\
&+\overset{3}{\Sigma} x_0^4 x_1 x_2 \cdot 2(2ab_0c_0 + ab_2g + ac_1h - 2afl - 2amn - a_1^2 c_1 - a_2^2 b_2 + 2a_1a_2f - 2a_1c_0h \\
&\qquad - 2a_2b_0g + 4a_1gm + 4a_2hn - 2a_1ln - 2a_2lm - 3ghl + 3l^3) \\
&+\overset{3}{\Sigma} x_0^3 x_1^3 \cdot 2(abn + ab_0f + ba_1g - 2ab_2m - 2ba_2l + 2a_2b_2h + 2a_1b_0n - a_1fh - b_0gh \\
&\qquad - 2a_1m^2 - 2b_0l^2 - 3h^2n + 6hlm) \\
&+\overset{6}{\Sigma} x_0^3 x_1^2 x_2 \cdot 2(abc_0 + 2ab_0c_1 - ba_2g - 3afm + 2a_1b_0c_0 + 4a_1b_2g - 2a_1c_1h + 2a_2b_0n \\
&\qquad - 4a_2b_2l + 5a_2fh - 3c_0h^2 - 2a_1fl - 6b_0gl - 2a_1mn - 2a_2m^2 + 3ghm + 6l^2m) \\
&+ x_0^2 x_1^2 x_2^2 (abc + 2ab_2c_1 + 2ba_2c_0 + 2ca_1b_0 - 3af^2 - 3bg^2 - 3ch^2 + 10a_1b_2c_0 + 10a_2b_0c_1 \\
&\qquad - 6a_1c_1m - 6a_2b_2n - 6b_0c_0l - 6a_1fn - 6a_2fm - 6b_0gn - 6b_2gl - 6c_0hm - 6c_1hl \\
&\qquad + 18fgh + 18lmn).
\end{aligned}
$$

The full development for S is

$$
\begin{aligned}
&\overset{3}{\Sigma} x_0^4 (ab_0c_0l - ab_0gn - ac_0hm + agm^2 + ahn^2 - almn - a_1a_2b_0c_0 + a_1^2 c_0 m + a_2^2 b_0 n \\
&\qquad + a_1b_0g^2 + a_2c_0h^2 - a_1c_0hl - a_2b_0gl - a_1^2 n^2 + a_1a_2mn - a_2^2 m^2 + a_1ghn + a_2ghm \\
&\qquad - 3a_1glm - 3a_2hln + 2a_1l^2n + 2a_2l^2m - g^2h^2 + 2ghl^2 - l^4) \\
&+\overset{6}{\Sigma} x_0^3 x_1 (abc_0l - abgn - ba_1a_2c_0 - ab_2c_0h + ab_0c_1l + ba_2^2 n - ab_0fg + ba_1g^2 + 2ab_2gm \\
&\qquad - ac_1hm - ba_2gl - ab_2ln + 2afhn - aflm + a_1^2 b_2 c_0 - a_1a_2b_0c_1 + a_2^2 b_0 f + a_2b_0c_0h \\
&\qquad + a_1^2 c_1 m + a_1a_2b_2n - 2a_2^2 b_2 m - a_1b_0c_0l + a_2b_2gh + a_2c_1h^2 - 2a_1^2 fn + a_1a_2fm \\
&\qquad + 2a_1b_0gn - 3a_1b_2gl - a_1c_1hl + 2a_2b_2l^2 - 2a_2b_0ln + a_1fgh - b_0g^2h - 3a_2fhl \\
&\qquad + 2a_1fl^2 - 2a_1gm^2 - a_2hmn + b_0gl^2 + a_1lmn + 2a_2lm^2 - gh^2n + 2ghlm \\
&\qquad + hl^2n - 2l^3m) \\
&+\overset{3}{\Sigma} x_0^2 x_1^2 (abc_0m + abc_1l - abfg - abn^2 - ab_0b_2c_0 - ba_1a_2c_1 + ab_2^2 g - ab_2c_1h + ba_2^2 f \\
&\qquad - ba_2c_0h + ab_0fn - ab_2fl + ba_1gn - ba_2gm + ab_2mn + ba_2ln + af^2h + bg^2h \\
&\qquad - afm^2 - bgl^2 + a_1^2 b_2 c_1 - a_2^2 b_2^2 + a_2 b_0^2 c_0 + a_1a_2b_2f + a_1b_2c_0h + a_2b_0b_2g + a_2b_0c_1h \\
&\qquad - a_1b_0c_0m - a_1b_0c_1l - a_1^2 f^2 - b_0^2 g^2 - a_1b_2gm - a_2b_0fl + 2a_2b_2hn + 2a_1b_0n^2 - 3a_1b_2ln \\
&\qquad - 3a_2b_0mn - a_1fhn - 2a_2fhm - b_0ghn - 2b_2ghl + 4a_1flm + 4b_0glm - a_1m^2n \\
&\qquad + 2a_2m^3 - b_0l^2n + 2b_2l^3 + fgh^2 - fhl^2 - ghm^2 - h^2n^2 + 4hlmn - 3l^2m^2)
\end{aligned}
$$

$$
\begin{aligned}
&+\overset{3}{\Sigma}x_0^2x_1x_2(abcl-bca_1a_2-abc_1g-acb_2h+ba_2^2c_1+ca_1^2b_2-2ab_0c_0f+ba_1c_0g\\
&\quad+ca_2b_0h+2ab_0c_1n+2ab_2c_0m-ba_2c_0l-ca_1b_0l+ab_2fg+ac_1fh-2ab_2n^2\\
&\quad-2ac_1m^2-af^2l+2afmn-a_1^2c_1f-a_2^2b_2f-a_1b_0c_1g-a_2b_2c_0h+2a_1b_0c_0n-a_1b_2c_0l\\
&\quad+2a_2b_0c_0m-a_2b_0c_1l+a_1a_2f^2+2a_1c_0fh+2a_2b_0fg-b_0c_0gh-a_1b_2gn-2a_1c_1hn\\
&\quad-2a_2b_2gm-a_2c_1hm-4a_1c_0m^2+5a_1c_1lm-4a_2b_0n^2+5a_2b_2ln-b_0c_0l^2+2b_2g^2h\\
&\quad+2c_1gh^2+a_1fgm+a_2fhn-2b_0g^2m-2c_0h^2n-3a_1fln-3a_2flm+4b_0gln\\
&\quad-2b_2gl^2+4c_0hlm-2c_1hl^2+2a_1mn^2+2a_2m^2n-4fghl-3ghmn+4fl^3\\
&\quad+4glm^2+4hln^2-7l^2mn).
\end{aligned}
$$

The leading coefficient of T may be obtained from Salmon's expression for the invariant T of the cubic, but the coefficients are too tedious to compute in general. For the coefficients of s and t see AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIX (1917), p. 232.

Note 2.

Certain invariants of the quartic have a well-defined geometrical meaning. The two simplest invariants, the A^3 and A^6, have apolarity meanings and may be included in the list by courtesy. Then comes the A^{15} of Professor Coble,* which is the condition that the quartic be reducible to the sum of the fourth powers of the six lines of a complete quadrilateral, also that the covariant S become two conics; the discriminant, A^{27}; the A^{48} of Dr. Thomsen, expressing that there be a polar conic which is the square of a line; an A^{54} of Professor Morley's expressing the condition that the quartic pass through the vertices of a pentagon; the undulation condition, A^{60}. In this list certain gaps are filled in by the A^{45}, the condition that a polar cubic be made up of three lines, and the A^{51}, the condition that the polar conic of two points be the square of their join.

There should also be an A^{24} under which the Steinerian has the stationary lines of the quartic as double lines. For the Steinerian in lines, an $A^{36}\xi^{18}$, and the Cayleyan, an $A^{12}\xi^{18}$, touch on the stationary lines.† Then the terms of the Steinerian containing only the first power of $(s\xi)^4$ or $(t\xi)^6$ should be the same as those of the Cayleyan, multiplied by an A^{24} to bring them up to the proper degree. If this A^{24} vanishes, then Σ has the stationary lines as double lines.

* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI (1909), p. 357.

† Proc. Nat. Ac. Sci., Vol. III (1917), p. 449.

On Surfaces Containing Two Pencils of Cubic Curves.

By C. H. Sisam.

1. In a recent article in the American Journal of Mathematics, Vol. XLI, p. 49, the author has discussed the surfaces generated by an algebraic system of cubic curves that do not constitute a pencil. In this article he studies the surfaces which contain two or more pencils of cubic curves.

2. Let m be the order of a given surface F which is generated by two pencils Σ_1 and Σ_2 of cubics which intersect in k variable points. We shall show that

$$mk \leq 18.$$

Any surface of order kx such that $kx > m$ contains all the curves of Σ_2, and hence has F as a component, if it contains $3x+1$ generic curves of Σ_1. Hence it contains F if it contains $3kx+1$ generic points on each of $3x+1$ generic curves of Σ_1, that is, if it satisfies at most, $(3x+1)(3kx+1)$ independent linear conditions. Then this number can not be less than the actual number of conditions that a surface of order kx must satisfy in order to contain F as a component, that is

$$(3x+1)(3kx+1) \geq \frac{(kx+3)!}{3!\,kx!} - \frac{(kx-m+3)!}{3!\,(kx-m)!},$$

or

$$3k(18-mk)x^2 + (3km^2 - 12km + 18k + 18)x - (m^3 - 6m^2 + 11m - 6) \geq 0.$$

Since this inequality holds for all integer values of x such that $xk > m$, we must have $18 - mk \geq 0$.

We have supposed, in the above proof, that F lies in three dimensions. But the theorem still holds if F belongs to $r > 3$ dimensions, since such a surface can be projected, without changing m or k, into one belonging to three dimensions.

Case I. The Cubics of Both Given Systems are Rational.

3. In this case the surface F is rational since it contains two pencils of rational curves. The entities Σ_1 and Σ_2, whose elements are the curves of the

two systems, are also rational since each defines a rational involution of order k on a generic curve of the other pencil.

$k=1$.

4. Let the pencils Σ_1 and Σ_2 be put in (1, 1) correspondence with pencils of lines in a plane Π having vertices at P_1 and P_2 respectively. Then the surface F and the plane Π are in (1, 1) correspondence in such a way that corresponding points are at the intersections of corresponding curves. Let μ be the order of the curves in Π that correspond to the hyperplane sections of F. Since these curves have points of multiplicity $\mu-3$ at P and P_1' we have $2(u-3)\leq\mu$ or $\mu\leq 6$.

5. The system of sextic curves in Π which have triple points at P and P' define parametrically a surface F which is of the given type. Any other such surface is a projection of this one, since any linear system of curves of order μ having $(\mu-3)$-fold points at P and P' is contained in the linear system of sextics. We conclude that if the curves of Σ_1 and Σ_2 are rational and $k=1$, *the surface is of order* 18, *belongs to an* S_{15}* *and is defined parametrically by the sextics in a plane which have two basis triple points or it is a projection (not necessarily from external points) of such a surface.*

6. If F belongs to S_{15}, the genus of a generic hyperplane section is equal to 4, that is, to the genus of a generic sextic with two triple points. It contains no cubic curves other than those of the two given systems.

$k=2$.

7. Let F be birationally transformed into the plane Π in such a way that the cubics of Σ_2 correspond to the lines through P'. Then the cubics of Σ_1 correspond to rational curves of order ν which have a $(\nu-2)$-fold point at P'. By a suitable birational transformation the order of the curves of this pencil can be reduced to 2. Let, then, $\nu=2$ and let P_1, P_2, P_3, P_4 be the basis points of this pencil of conics. To the hyperplane sections of F correspond curves of order μ which have a $(\mu-3)$-fold point at P' and multiplicities ρ_i at P_i, $(i=1, 2, 3, 4)$ such that $2\mu-(\rho_1+\rho_2+\rho_3+\rho_4)=3$. By a suitable birational transformation, such a system of curves can be reduced to one of the following three types:

(1) $\mu=4$ $\rho_1=2$ $\rho_2=\rho_3=\rho_4=1$.
(2) $\mu=3$ $\rho_1=0$ $\rho_2=\rho_3=\rho_4=1$.
(3) $\mu=3$ $\rho_1=2$ $\rho_2=1$, $\rho_3=\rho_4=0$.

* The symbol S_r denotes a space of r dimensions.

8. The surfaces defined parametrically by systems of types (2) and (3) are projections of special cases of surfaces of type (1) such that the latter surface has one or more double points. Hence, if the cubics of Σ_1 and Σ_2 are rational and $k=2$, *the surface is of order* 8, *belongs to an* S_7 *and is defined parametrically by the quartics in a plane which have a double and four simple basis points, or it is a projection of such a surface.*

9. If F belongs to S_7, its generic hyperplane sections are of genus 2. It contains four pairs of pencils of cubics such that cubics of opposite systems intersect in two points. One pencil of such a pair is defined by the pencil of conics through the double and three simple basis points. The other pencil is defined by the lines through the remaining basis points. It contains no other cubic curves.

$k=3.$

10. As in the preceding case, this surface can be transformed into a plane Π in such a way that the cubics of Σ_2 correspond to the lines through P' and the cubics of Σ_1 to the cubics which have a basis double point P_1 and five basis simple points $P_2, P_3, \ldots, P_6$. To the hyperplane sections of F correspond curves of order μ which have a $(\mu-3)$-fold point at P' and multiplicities ρ_i at P_i, $(i=1, 2, 3, \ldots, 6)$ such that $3\mu-(2\rho_1+\rho_2+\rho_3+\rho_4+\rho_5+\rho_6)=3$. If the order m of F exceeds three, these curves can be transformed into quartics which have a double point at P_1 and pass simply through $P_2, P_3, \ldots, P_6$ and P'. If $m\leq 3$ the surface is a projection of one with one or more singular points defined by a system of the above type. Hence, if the cubics of Σ_1 and Σ_2 are rational and if $k=3$, *the surface is of order* 6, *it belongs to an* S_6 *and is defined parametrically by the quartics in a plane which have a double and six simple basis points, or it is a projection of such a surface.*

11. There are thirty-two pencils of cubic curves on such a surface F_1 defined by the pencils of curves of order $i\leq 3$ in Π which have an $(i-1)$-fold point at P_1 and pass through $2i-1$ simple basis points. The curves of each pencil intersect those of one other pencil in three points. The S_3 defined by the curves of such a pair of pencils constitute the two systems of S_3 on an hyperquadric in S_5 which has a double line and contains F.

There are twelve other cubic curves on F. Six are defined by the conics through five simple basis points and the rest by the cubics which have a node at one simple basis point and pass through all the other basis points.

$k\geq 4.$

12. If $k=4$ and if $m=4$, the surface F can be transformed into Π in such a way that the curves Σ_2 correspond to the lines through P', the curves

Σ_1 to quartics with a basis double point P_1 and seven basis simple points $P_2, P_3, \ldots, P_8$, and that the plane sections of F correspond to quartics which have a double point at P_1 and pass through $P_2, P_3, \ldots, P_8$ and P'. Such a linear system defines a quartic F with a double line.

We have seen (Art. 2) that if $k>4$, then $m\leq 3$. Hence, *if* $k\geq 4$, *the surface F is a quartic with a double line, or a cubic, a quadric or a plane.*

CASE II. THE CUBICS OF Σ_1 ARE RATIONAL; THOSE OF Σ_2, ELLIPTIC.

$k=1$.

13. The pencil Σ_2 is rational since it is in (1, 1) correspondence with a generic curve of Σ_1. Similarly, Σ_1 is elliptic. Let the pencil Σ_1 be put in (1, 1) correspondence with the pencil of rectilinear generators of a ruled quintic surface ϕ belonging to S_4 and let the pencil Σ_2 be put in (1, 1) correspondence with a pencil Σ' of quartics on ϕ having three basis points P_1, P_2, P_3. Then F and ϕ are in (1, 1) correspondence in such a way that corresponding points are at the intersections of corresponding curves.

14. Denote by ρ_1, ρ_2, ρ_3 the multiplicities at P_1, P_2, P_3 of the curves on ϕ which correspond to the hyperplane sections of F. The curves of this linear system intersect the rectilinear generators in three points and the quartics of Σ' in $3+\rho_1+\rho_2+\rho_3$ points. Their order $6+\rho_1+\rho_2+\rho_3$ is found by counting their intersections with an hyperplane containing a quartic of Σ' and a generator. These curves are cut from ϕ by a linear system of quintic hypersurfaces H^5 which have contact of second order with ϕ at P_1, P_2 and P_3, and contain as basis curves (a), the generators through P_1, P_2, P_3 counted respectively $3-\rho_1, 3-\rho_2, 3-\rho_3$ times, (b) a quintic curve C^5 which intersects the generators twice and (c), five generators of ϕ. For, since C^5 is elliptic, in a linear system of seventy-four dimensions* of H^5 that do not contain ϕ, there exists a linear system of thirty-one dimensions of H^5 which contain C^5 and have contact of second order at P_1, P_2 and P_3. The curves corresponding to the hyperplane sections of F intersect these H^5 in $3(\rho_1+\rho_2+\rho_3)$ points at P_1, P_2, P_3; in $2(\rho_1+\rho_2+\rho_3)-3$ points on C^5 and in thirty-three other points. These curves are of genus 3 (at least) since F is neither ruled nor rational,† hence each curve of the system lies on such an H^5. The residual intersection does not intersect a generic generator. Hence it degenerates into the generators through P_1, P_2, P_3 counted $3-\rho_1, 3-\rho_2$ and $3-\rho_3$ times and five generators.

* Cf. the Author, AMERICAN JOURNAL OF MATHEMATICS, Vol. XLI, p, 51.

† Cf. Castelnuovo-Enriques, *Mathematische Annalen*, Vol. XLVIII, p. 308.

The linear system of curves on ϕ which corresponds to the hyperplane sections of F is thus contained in (or coincides with) a linear system L of curves of order 15 and genus 7, having triple points at P_1, P_2, P_3 which is cut from ϕ by a linear system of H^5 which contain C^5 and five fixed generators, and have contact of second order with ϕ at P_1, P_2, P_3. It follows that: *the surface F is of order* 18, *it belongs to a space of eleven dimensions and its generic hyperplane sections are of genus* 7, *or it is a projection of such a surface.*

15. Let the correspondence between Σ_1 and the generators of ϕ be set up in such a way that the g_3^2 defined on a given generic cubic C of Σ_2 corresponds to the g_3^2 defined on the quartic corresponding to C by the hypercubics having second order contact with ϕ at P_1, P_2, P_3. Then the linear system L is cut from ϕ by this system of hypercubics, that is: *the surface F can be birationally represented on a ruled quintic belonging to S_4 in such a way that the curves corresponding to the hyperplane sections of F are cut from the quintic by cubic hypersurfaces which have second order contact with the quintic at the basis points of a pencil of quartic curves on it.*

16. Let F belong to S_{11}. Its parametric equations can be reduced to the form

$$\begin{array}{llll} x_0=1 & x_1=v & x_2=v^2 & x_3=v^3 \\ x_4=\wp(u) & x_5=v\wp(u) & x_6=v^2\wp(u) & x_7=v^3\wp(u) \\ x_8=\wp'(u) & x_9=v\wp'(u) & x_{10}=v^2\wp'(u) & x_{11}=v^3\wp'(u), \end{array}$$

wherein $\wp(u)$ is the Weierstrassian $\wp$-function.

The residual section of F by an hyperplane which contains the planes of three cubics of Σ_2 degenerates into three curves of Σ_1 since it intersects a generic curve of Σ_2 in three points and has no points in common with a generic curve of Σ_1. Three such cubics of Σ_1 lie in the S_7 defined by the S_3 of two of them, since every hypersurface that contains two of them contains the third.

17. The order of any curve on F that intersects the curves of Σ_1 in x points, and those of Σ_2 in y points is

$$n=3(x+y),$$

as may be seen by counting its intersections with a generic hyperplane which intersects F in three cubics of each system. It follows that the order of every curve on F is a multiple of three. Moreover, since there are no rational curves on F except those of Σ_1, and since every curve C on F for which $y=1$ is in (1, 1) correspondence with Σ_2 and hence is rational, we conclude that *there are no curves of order less than nine on the surface F belonging to S_{11}*

except those of Σ_1 and Σ_2. Curves of order 9 on F actually exist. Such a curve is defined by a generic cubic on ϕ, or by a generic quartic through P_1, P_2 or P_8.

$$k=2.$$

18. The involution γ_2^1 of order 2 defined by Σ_1 on a generic cubic C of Σ_2 is rational. For, if γ_2^1 is elliptic, it has no double points, so that no curve of Σ_1 has its two intersections with C coincident. But the involution defined by Σ_2 on a generic curve of Σ_1 has two double points. Hence each of two fixed curves of Σ_2 would degenerate into a curve counted twice. These degenerate curves are of genus 1 since they are in (1, 1) correspondence with Σ_1. But this is impossible, since the curves of Σ_2 are cubics. Then γ_2^1 is rational, Σ_1 is rational, and F is rational since it contains a rational pencil of rational curves.

19. The surface F can be transformed into a plane Π in such a way that to Σ_1 corresponds a pencil of lines through P' and to Σ_2, a pencil of cubics through P' and eight other points $P_1, P_2, \ldots, P_8$. The linear system of curves of order μ corresponding to the hyperplane sections of F can be reduced to a system which coincides with, or is contained in the complete linear system of quartics which pass through P' and have multiplicities ρ_i at P_i such that either

$$(1) \quad \rho_1=\rho_2=\ldots=\rho_8=1,$$

or

$$(2) \quad \rho_1=2,\ \rho_2=\ldots=\rho_7=1,\ \rho_8=0.$$

Hence, if a generic cubic of Σ_1 is rational, and of Σ_2 elliptic, and if $k=2$, then either

(1) *The surface is of order 7, it belongs to an S_5 and is defined parametrically by the quartics in a plane through the nine basis points of a pencil of cubics, or it is a projection of such a surface,* or (2) *The surface is of order 5, it belongs to an S_4 and is defined parametrically by the quartics in a plane with a double and seven simple basis points, or it is a projection of such a surface.*

On the surface (1), there are in general eight other pencils of rational cubics which intersect the cubics of Σ_2 in two points and 126 cubics which intersect the cubics of Σ_2 once. On the surface (2) there are in general, sixty-three other pencils of rational cubics which intersect the cubics of Σ_2 twice, and eighty-six cubics which intersect those of Σ_2 once.

$$k=3.$$

20. As in the preceding case, the involution γ_3^1 defined by the curves of Σ_1 on a generic curve of Σ_2 is rational. For, if not, it has no double points,

while the involution defined by Σ_2 on a generic curve of Σ_1 has six double points. Then six curves of Σ_2 have as double component a curve of genus 1, which is impossible.

It follows that F is rational. It can be represented on a plane Π in such a way that Σ_1 corresponds to a pencil of lines through P', Σ_2 to a pencil of cubics through $P_1, P_2, \ldots, P_9$ and the hyperplane sections to a system of curves which coincide with, or are contained in the complete system of quartics which pass through P' and have multiplicities ρ_i at P_i such that either

$$(1) \quad \rho_1=\rho_2=\ldots=\rho_9=1,$$

or

$$(2) \quad \rho_1=2,\ \rho_2=\rho_3=\ldots=\rho_8=1,\ \rho_9=0.$$

Hence, if a generic cubic of Σ_1 is rational and of Σ_2, elliptic, and if $k=3$, then either

(1) *The surface is of order* 6, *belongs to an* S_4 *and is defined parametrically by the quartics through ten points of which nine are the basis points of a pencil of cubics, or it is a projection of such a surface,* or (2) *The surface is a quartic with a nodal line. It belongs to* S_3.

21. Each of these surfaces is the projection of one of the surfaces of Art. 19 from a point P on the surface. Each has a double line l which is the projection of the elliptic cubic through P. The cubics of Σ_1 are the projections of a pencil of quartics through P. On the surface (1), the cubics of Σ_1 have l as a locus of double points. The planes of the cubics of Σ_1 and Σ_2 on this surface in S_4 generate a singular hyperquadric which has F as its complete intersection with an hypercubic.

CASE II. THE CUBICS OF BOTH SYSTEMS ARE ELLIPTIC.

$k=1$.

22. We can write at once the parametric equations of a surface which clearly belongs to the required type. Let $\wp(u)$ and $\bar{\wp}(v)$ be two Weierstrassian $\wp$-functions (having in general different moduli). Then the surface

$$\left.\begin{array}{lll} x_0=1 & x_1=\wp(u) & x_2=\wp'(u) \\ x_3=\bar{\wp}(v) & x_4=\wp(u)\bar{\wp}(v) & x_5=\wp'(u)\bar{\wp}(v) \\ x_6=\bar{\wp}'(v) & x_7=\wp(u)\bar{\wp}'(v) & x_8=\wp'(u)\bar{\wp}'(v) \end{array}\right\} \qquad (1)$$

is generated by two pencils of elliptic cubics $u=$const. and $v=$const. such that cubics of opposite systems intersect in one point.

23. Any two cubics of one system on (1) lie in an S_5 which contains a third cubic of the same system. The residual section of the surface by an

hyperplane that contains such an S_5 is composed of three cubics of the opposite system. Hence, the order of this surface is 18 and, since cubics of the same system do not intersect, the genus of a generic hyperplane section is equal to 10.* It follows further that a curve on the surface which intersects the curves $u=\text{const.}$ in x-points and the curves $v=\text{const.}$ in y-points is of order $n=3(x+y)$ as may be seen by counting the intersections of the curve with an hyperplane which intersects (1) in six cubics. If either $x=1$, $y>0$ or $y=1$, $x>0$ there is a $(1, y)$ or $(1, x)$ correspondence between the two systems of cubics, and hence a particular relation between the moduli. Hence, *in general there are no curves of order less than* 12 *on the surface* (1) *except the cubics* $u=$*const. and* $v=$*const.*

24. Let F be a given surface generated by two pencils Σ_1 and Σ_2 of elliptic cubic curves that intersect in one point. Let the moduli of the pencils on (1) be respectively equal to those of Σ_1 and Σ_2 and let the pencils $u=\text{const.}$ and Σ_1 and $v=\text{const.}$ and Σ_2 be put in (1, 1) correspondence. Then the surfaces (1) and F are in (1, 1) correspondence in such a way that corresponding points are at the intersection of corresponding curves. To an hyperplane section of F corresponds on (1) a curve C^{18} which is of order 18 since it intersects the curves $u=\text{const.}$ and $v=\text{const.}$ in three points.

25. In the S_8 defined by the surface (1), there exists a linear system of $9n^2-1$ dimensions of hypersurfaces H^n of order n which do not contain the surface (1) since such an H^n can be made to contain $3n-1$ generic curves $u=\text{const.}$ and $3n-1$ generic points of an additional curve $u=\text{const.}$ without containing (1). If n is sufficiently large, there are, in such a linear system, at least $3n-12$ linearly independent H^n which contain a given C^{18} and $3n-4$ given generic cubics $u=\text{const.}$ The residual intersection of all these H^n with any given generic cubic $v=\text{const.}$ is a fixed point P. Hence they all contain a fixed curve C on (1) which intersects the cubics $v=\text{const.}$ in one point and the cubics $u=\text{const.}$ in a certain number $y\geq 0$ of points. Their residual intersections with (1) have no points in common with a generic curve $v=\text{const.}$ and break up into $3n-3-y$ curves of that pencil. The curve C is of order $3(1+y)$. Its genus is unity since it intersects the curves $v=\text{const.}$ once. In general, $y=0$ (Art. 23).

26. If $y=0$, the curve C is a cubic of $u=\text{const.}$ and the curves C^{18} corresponding to the hyperplane sections of F are defined on (1) by a linear system of H^n through $3n-3$ fixed cubics of each system. The complete linear system to which they belong is of dimension $9n^2-1-[3n(3n-3)+3(3n-3)]=8$.

* Cf. Castelnuovo-Enriques, *Annale de Matematica*, Ser. 3, Vol. VI, p. 171.

Let
$$u_1, u_2, \ldots, u_{3n-3} \text{ and } v_1, v_2, \ldots, v_{3n-3}$$
be the parameters of the $6n-6$ basis cubics of the given system of H^n. The parameters u, u', u'' of the three intersections with any cubic $v=\text{const.}$ of a C^{18} corresponding to an hyperplane section of F satisfy the congruence

$$u+u'+u''\equiv-(u_1+u_2+\ldots+u_{3n-3}) \quad \text{mod } (\omega_1\omega_2).$$

Similarly, the parameters v, v', v'' of its intersections with a cubic $u=\text{const.}$ satisfy

$$v+v'+v''\equiv-(v_1+v_2+\ldots+v_{3n-3}) \quad \text{mod } (\overline{\omega}_1\overline{\omega}_2).$$

Let the correspondence between F and (1) be set up in such a way that the g_3^2 defined by the lines in the plane of one given cubic of each system on F is transformed into the g_3^2 defined by the lines in the plane of the corresponding cubic on (1). Then each member of the above congruences is congruent to zero, and the complete linear system of eight dimensions, to which the curves C^{18} belong is cut from (1) by a linear system of H^n which degenerate into a fixed H^{n-1} containing the $6n-6$ cubics, together with the system of hyperplanes of S_8. We conclude that *in general, $y=0$ and the surface F coincides with a surface* (1) *or with a projection of such a surface.*

27. If $y>0$, the curves corresponding to the hyperplane sections of F are cut from (1) by a linear system of H^n which contain $3n-4$ fixed cubics $u=\text{const.}$, $3n-3-y$ fixed cubics $v=\text{const.}$, and a fixed curve C of order $3(1+y)$ and genus 1. The complete linear system L of curves cut from (1) by the H^n which contain the above curves is of dimension

$$9n^2-1-[3(1+y)n+(3n-4)(3n-y)+(3n-3-y)3]=8-y.$$

Since the dimension of this system can not be less than 3, we have $y\leq 5$.

The curve C intersects the curves of the system L in $3+5y$ points. In fact, by varying the basis curves $u=\text{const.}$ we can obtain a pencil of curves C (no two of which intersect) each of which, with fixed curves $u=\text{const.}$ and $v=\text{const.}$, forms the system of basis curves of a linear system of H^n that define L. Each curve C intersects an H^n of a system which does not contain it in $3n(1+y)$ points of which $3n(1+y)-5y-3$ lie on the basis cubics and $5y+3$ on a curve of L. It follows that the order of the system L and hence of the surface F, is

$$m=18n-(5y+3)-3(6n-7-y)=18-2y.$$

Let p be the genus of a generic curve of L. The virtual genus of the residual section of (1) by an H^n of a linear system which defines L is

$9(n-1)^2-y+1$, and of the complete section of (1) by H^n is $9n^2+1$. Then

$$p+9(n-1)^2-y+1+18n-(18-2y)-1=9n^2+1$$

or

$$p=10-y.$$

Hence, *if* $5 \geq y \geq 0$, *the surface* F *is of order* $18-2y$, *its hyperplane sections are of genus* $10-y$ *and it belongs to a space of* $8-y$ *dimensions, or it is a projection of such a surface.*

$$k=2.$$

28. If F belongs to an $S_r (r>3)$ let it be projected from an S_{r-4} which does not intersect it onto a surface F' belonging to S_3. Denote the projections of Σ_1 and Σ_2 by Σ_1' and Σ_2'. If the involution γ_2^1 defined on a generic curve C' of Σ_1' by the curves of Σ_2' is elliptic, the lines joining corresponding points envelope a curve of class 3, if it is rational, they constitute a pencil with vertex on C'. Since at most one cubic of Σ_2 can be coplanar with C'—otherwise a generic cubic of Σ' would have four points in common with this plane—the surface F' belongs to one of the following types:

α. The curves of Σ_1' and Σ_2' lie in pairs in the tangent planes of a developable of class 4 and genus 1. This developable can not reduce to a cone since it can not have a double plane.

β. The planes of each system envelope a cone of class 3 and genus 1.

γ. The planes of Σ_1' generate an axial pencil; those of Σ_2' envelope a cone of class 3 and genus 1.

δ. The planes of each system generate an axial pencil.

ε. The curves of Σ_1' and Σ_2' lie in pairs in the tangent planes to a quadric cone.

29. Cases α, β and γ do not exist. In case α, the residual section of F' by a plane of the developable is elliptic since it is intersected by the curves of Σ_1 and Σ_2 in one variable point. It is a cubic since the order of F' does not exceed 9 (Art. 2). Denote this system of cubics by Σ_3'. Two generic cubics of Σ_3' intersect in one variable point since, of the three intersections of each with the plane of the other, only two lie on the curves of Σ_1' and Σ_2'. Hence F' contains a pencil of rational curves* on which the pencil Σ_1' (or Σ_2') defines an elliptic involution. This is impossible.

In case β, let P_1 and P_2 be the vertices of the cones. The planes of three curves of Σ_2' pass through P_1. The system of lines joining corresponding points of the involution on each of these cubics degenerates into a pencil counted three times. Then these curves degenerate and intersect a generic

* Cf. the Author, AMERICAN JOURNAL OF MATHEMATICS, Vol. XLI.

curve of Σ_1 in coincident points. But this is impossible since the involution on a generic curve of S_1 has no double points.

In case γ, let μ be the multiplicity on F' of the axis l of the pencil of planes of Σ'_1. Then the order of F' is $\mu+3$. A generic curve C' of Σ'_2 intersects l. The residual section in its plane is of order μ and has a $(\mu-1)$-fold point on l. Hence it is rational. The curves of Σ'_2 define an elliptic involution on this rational curve, which is impossible. We now conclude that Σ'_1 and Σ'_2 are both rational.

30. In case δ, we denote the axes of the pencils by l_1 and l_2 and suppose, first, that they do not intersect. Then l_1 (or l_2) is a simple line on the surface since a generic point of it is a simple point on just one curve of Σ'_2 (or Σ'_1). Hence F' is a quartic with two skew simple lines. Since l_1 and l_2 do not intersect, and the cubics of Σ_1 and Σ_2 are elliptic, the quartic F is not the projection of a surface of the same order belonging to more than three dimensions; for, the genus of a generic plane section is less than 2 only if the surface is a ruled quartic of genus 1.

31. If l_1 and l_2 intersect, all the curves of both systems pass through their intersection. Let μ be the multiplicity of l_1 and l_2 on F'. Then one curve of each system degenerates into the axis of the other system counted μ times, together with a residual curve of order $3-\mu$. Thus $\mu\leq 3$ and $m=3+\mu\leq 6$. The intersection of l_1 and l_2 is a $(\mu+1)$-fold point on the surface. The linear system of quadrics which contain l_1 and l_2 defines a birational transformation of F' into a sextic surface F_1 belonging to S_4 and of Σ'_1 and Σ'_2 into pencils of cubics on F_1, all of which pass through a fixed point P which is a double point on F_1. The planes of these two pencils of cubics constitute the two systems of planes on an hyperquadric H^2 with a double point at P. The surface F_1 lies on (at least) six linearly independent hypercubics of which at most five have H^2 as a component. Hence, it forms the complete intersection of H^2 with an hypercubic. The surface F' is the projection of F_1 from a point on H^2.

32. In case ε, all the cubics of both systems pass through the vertex of the cone which is a double point on F'. The order of F' is 6 since a residual curve in the plane of two coplanar cubics would have to be a component of a curve of each system. Three of the intersections of coplanar cubics lie on a generator of the quadric cone. The other six lie on a nodal sextic which forms the intersection of a cubic and a quadric surface. The linear system of cubic surfaces which contain this nodal sextic define a birational transformation of

F' into a surface of the type of F_1 of Art. 31. The surface F' is the projection of the surface F_1 from a point not lying on the hyperquadric.

33. Every surface F belonging to S_4 is of the type F_1 since it lies on, at most, one hyperquadric, and its projection from a point not on this hyperquadric is a sextic F' of the type of Art. 32. Since no surface F_1 is a projection of a surface of the same order belonging to S_5, we conclude that, if generic cubics of Σ_1 and Σ_2 are elliptic and if $k=2$, then

(1) *If the cubics of both systems do not all pass through a fixed point P, the surface is a quartic, belongs to S_3 and has two skew simple lines.*

(2) *If the cubics of both systems pass through a fixed point P, the surface is the complete intersection of a cubic hypersurface in S_4 with a quadric hypersurface which has a double point at P, or it is a projection of such a surface.*

34. Neither of these surfaces contains, in general, any other pencils of cubics. In fact, their geometric genus is in general unity so that they do not contain a pencil of rational cubics. If the surface (1) contains an additional pencil of elliptic cubics, it contains an additional simple line, if the surface (2) belonging to S_4 contained such a pencil, the planes of the cubics would lie on the hyperquadric.

$k=3$.

35. If F belongs to an S_r $(r>3)$ let it be projected from an S_{r-4} that does not intersect it into a surface F' belonging to S_3. The lines of section of the plane of a cubic C' of Σ_1' by the the planes of Σ_2' constitute a pencil since through a generic point of C' there passes just one such line. Hence, either the planes of Σ_2' (and, similarly, of Σ_1') constitute an axial pencil or the curves of the two systems lie in pairs in the tangent planes to a quadric cone.

36. Let the two systems of planes constitute two pencils whose axes l_1 and l_2 do not intersect. Then neither l_1 nor l_2 lies on F' so that F' is a cubic. Since a generic curve of Σ_1 (or Σ_2) is elliptic, this cubic can not have a nodal line and is not the projection of a surface of the same order belonging to S_4.

37. If the axes of the pencils intersect, or if the curves lie in pairs in the tangent planes to a quadric cone, we see as in Arts. 31 and 32, that F' is the projection of a surface F_1 belonging to S_4 which forms the complete intersection of a cubic hypersurface with a quadric hypersurface having a double point which does not lie on the cubic.

38. It follows as in Art. 33, that if generic cubics of Σ_1 and Σ_2 are elliptic and if $k=3$, then

(1) *If the planes of the cubics of both systems do not all pass through the same fixed point, the surface is a cubic which can not have a double line.*

(2) *If the planes of the cubics of both systems pass through the same fixed point, the surface is the complete intersection of a cubic hypersurface in S_4 with a quadric hypersurface which has a double point not lying on the cubic, or it is a projection of such a surface.*

As in Art. 34, it is seen that the surface (2) does not in general contain any other pencils of cubic curves.

39. The results of the foregoing discussion are summarized in the following table. In this table p_1 and p_2 denote respectively the genus of a generic curve of Σ_1 and Σ_2, k is the number of variable intersections of curves of opposite systems; m is the maximum order of a surface F containing Σ_1 and Σ_2; p is the maximum genus of an hyperplane section of F; r is the maximum number of dimensions to which F belongs, and s is the number of distinct or consecutive pencils of cubic curves necessarily existing on the surface F of maximum order. The numbers p_g and p_a are respectively the geometric and arithmetic genus of F except in the last two cases, where they are the maximum values of those numbers.

p_1	p_2	k	m	r	p	s	p_g	p_a
0	0	1	18	15	4	2	0	0
0	0	2	8	7	2	8	0	0
0	0	3	6	5	2	32	0	0
0	0	4	4	3	2	128	0	0
0	1	1	18	11	7	2	0	—1
0	1	2	7	5	3	10	0	0
0	1	3	6	4	3	11	0	0
1	1	1	18	8	10	2	1	—1
1	1	2	6	4	4	2	1	1
1	1	3	6	4	4	2	1	1

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Modular Invariants of a Quadratic Form for a Prime Power Modulus.

By J. E. McAtee.*

Introduction.

We consider a quadratic form q in n variables whose coefficients are integers taken modulo P^λ, where P is an odd prime. Two such forms are said to be equivalent modulo P^λ if they become identically congruent under a linear homogeneous transformation whose coefficients are congruent to integers modulo P^λ and their determinant is congruent to unity. All the forms equivalent to q are said to form a class. A single-valued function ϕ of the coefficients of q is an invariant of q if it has the same value for all forms of the same class. In case the values taken by ϕ are integers which may be reduced modulo P^λ, the invariant is called modular. We obtain a complete invariantive classification of n-ary quadratic forms modulo P^λ and construct modular invariants which completely characterize the classes. Professor Dickson † had treated the problem for the case $\lambda=1$, when the fundamental invariants are the determinant D of q, the rank r modulo P of D, and A_r, to which the invariant $A_{\bar{R}_0}(D)$ of § 4 reduces for $\lambda=1$.

In his investigation of the number of sets of solutions of the congruence $q\equiv c \pmod{P^\lambda}$, C. Jordan ‡ obtained from q by a seriatim process of reduction the canonical form

$$P^{e_1}(A_1x_1^2+\ldots+A_sx_s^2)+P^{e_2}(A_{s+1}x_{s+1}^2+\ldots+A_tx_t^2)+P^{e_3}()+\ldots,$$

but gave no explicit method to deduce à priori from q the number of terms in each parenthesis or the values of $e_1, e_2 \ldots$. This is accomplished by the present invariants.

The second part of the paper is devoted to the polynomial modular invariants of a binary quadratic form modulo P^λ, in particular to the determination of a fundamental system of such invariants modulo 4.

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* Dr. McAtee died at Urbana, Ill., Dec. 2, 1918. The proof-sheets were compared with both the final and an initial draft of the thesis. In § 7, an inadequate proof of Theorem 1 has been replaced by the present shorter proof, while the easy proofs of some other facts have been omitted. The proof of formula (3) of § 1 has been suppressed, as it was entirely similar to that cited. L. E. Dickson.

† Madison Colloquium Lectures, 1913, Ch. 1; *Trans. Amer. Math. Soc.*, Vol. X (1909), p. 123.

‡ *Jour. de Math.*, (2), Vol. XVII (1872), pp. 368–402. Cf. Bachmann, "Die Arith. der Quadratischen Formen" (1898), p. 434.

I. Invariants of the n-ary Quadratic Form Modulo P^λ.

1. *Canonical Forms and Classes.*—Consider the form

$$q_n=\sum_{i,j=1}^{n} a_{ij}x_ix_j \qquad (a_{ij}=a_{ji}), \tag{1}$$

where the a_{ij} are integers taken modulo P^λ, where P is an odd prime. If every a_{ij} is divisible by P^λ, the form (1) is said to constitute the class C_0. In the contrary case, let e_1 be the exponent of the highest power of P that divides every a_{ij}, and write

$$q_n=P^{e_1}\sum_{i,j=1}^{n} A_{ij}^{(1)}x_ix_j, \qquad D=|A_{ij}^{(1)}|. \tag{2}$$

First, let the determinant D be prime to P. Since there exists a primitive root of P^λ, it follows as in the Madison Colloquium Lectures, pp. 6, 7, that q_n can be transformed into

$$P^{e_1}[x_1^2+\ldots\ldots+x_{n-1}^2+Dx_n^2] \tag{3}$$

by a linear transformation with integral coefficients of determinant unity modulo P^λ.

Second, let D be divisible by P, and r_1 its rank modulo P. As on p. 9 of the Lectures cited, we may assume that

$$M_{r_1}=|A_{ij}^{(1)}|\not\equiv 0 \pmod P \qquad (i,j=1,\ldots,r_1),$$

and show* that q_n can be transformed into

$$P^{e_1}[\sum_{i,j=1}^{n-1} A_{ij}^{(1)}x_ix_j+2\sum_{j=r_1+1}^{n-1} B_{jn}^{(1)}x_jx_n+B_{nn}^{(1)}x_n^2],$$

where

$$B_{jn}^{(1)}\equiv\begin{vmatrix} A_{11}^{(1)}\ldots A_{1r_1}^{(1)} & A_{1n}^{(1)} \\ \ldots\ldots\ldots\ldots & \\ A_{r_11}^{(1)}\ldots A_{r_1r_1}^{(1)} & A_{r_1n}^{(1)} \\ A_{j1}^{(1)}\ldots A_{jr_1}^{(1)} & A_{jn}^{(1)}\end{vmatrix} M_{r_1}^{-1} \qquad (j=r_1+1,\ldots,n)$$

is congruent to an integer modulo P^λ and is divisible by P. Repetitions of this process give

$$P^{e_1}[\sum_{i,j=1}^{r_1} A_{ij}^{(1)}x_ix_j+\sum_{i,j=r_1+1}^{n} B_{ij}^{(1)}x_ix_j], \tag{4}$$

where

$$B_{ij}^{(1)}\equiv\begin{vmatrix} A_{11}^{(1)}\ldots A_{1r_1}^{(1)} & A_{1j}^{(1)} \\ \ldots\ldots\ldots\ldots & \\ A_{r_11}^{(1)}\ldots A_{r_1r_1}^{(1)} & A_{r_1j}^{(1)} \\ A_{i1}^{(1)}\ldots A_{ir_1}^{(1)} & A_{ij}^{(1)}\end{vmatrix} M_{r_1}^{-1}$$

is congruent to an integer modulo P^λ and is divisible by P. If now all the $P^{e_1}B_{ij}^{(1)}$ are zero modulo P^λ, we have q_n replaced by

$$P^{e_1}\sum_{i,j=1}^{r_1} A_{ij}^{(1)}x_ix_j.$$

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* Using $P^{e_1}B_{jn}\equiv 0 \pmod{P^\lambda}$ in place of $B_{jn}\equiv 0 \pmod P$.

In the contrary case let e_2 denote the highest power of P that divides all the $P^{e_1}B_{ij}^{(1)}$ $(e_1<e_2<\lambda)$. Then we have (4) expressed in the form

$$P^{e_1}\sum_{i,j=1}^{r_1} A_{ij}^{(1)}x_ix_j+P^{e_2}\sum_{i,j=r_1+1}^{n} A_{ij}^{(2)}x_ix_j, \tag{5}$$

where at least one of the $A_{ij}^{(2)}$ is prime to P. The question now arises as to whether a different choice of M_{r_1} might not lead to a set of $B_{ij}^{(1)}$ that would give a different e_2. To answer this we note that, since M_{r_1} is prime to P and hence is congruent to a power ρ^m of a primitive root ρ of P^λ, we can transform (5) into

$$P^{e_1}[x_1^2+\ldots\ldots+x_{r_1-1}^2+\rho^m x_{r_1}^2]+P^{e_2}\sum_{i,j=r_1+1}^{n} A_{ij}^{(2)}x_ix_j.$$

By applying a transformation of the type

$$x_{r_1}=\rho^{-k}x'_{r_1},\quad x_n=\rho^k x'_n,$$

we obtain one of the forms

$$\left.\begin{array}{ll} P^{e_1}[x_1^2+\ldots\ldots+x_{r_1}^2]+P^{e_2}q & (q \text{ free of } x_1,\ldots\ldots,x_{r_1}), \\ P^{e_1}[x_1^2+\ldots\ldots+x_{r_1-1}^2+\rho x_{r_1}^2]+P^{e_2}q_\rho & (q_\rho \text{ free of } x_1,\ldots\ldots,x_{r_1}). \end{array}\right\} \tag{6}$$

If we had chosen M'_{r_1} instead of M_{r_1}, we would have obtained from (2) one of

$$\left.\begin{array}{ll} P^{e_1}[y_1^2+\ldots\ldots+y_{r_1}^2]+P^{e'_2}q' & (q' \text{ free of } y_1,\ldots\ldots,y_{r_1}), \\ P^{e_1}[y_1^2+\ldots\ldots+y_{r_1-1}^2+\rho y_{r_1}^2]+P^{e'_2}q'_\rho & (q'_\rho \text{ free of } y_1,\ldots\ldots,y_{r_1}). \end{array}\right\} \tag{7}$$

Since* $\qquad x_1^2+\ldots\ldots+x_{r_1}^2$ and $y_1^2+\ldots\ldots+y_{r_1-1}^2+\rho y_{r_1}^2$

are not equivalent modulo P, it is evident that if we obtain (6_1) by the use of M_{r_1} we must obtain (7_1) by the use of M'_{r_1}. Similarly, if we obtain (6_2) by the use M_{r_1} we must obtain (7_2) by the use of M'_{r_1}. For definiteness, suppose that we obtain (6_1) and (7_1) and that $e_2<e'_2$. Then the forms

$$P^{e_1}(x_1^2+\ldots\ldots+x_{r_1}^2)+P^{e_2}q \text{ and } P^{e_1}(y_1^2+\ldots\ldots+y_{r_1}^2)$$

are transformable into each other modulo $P^{e'_2}$, whenever the forms

$$x_1^2+\ldots\ldots+x_{r_1}^2+P^k q \text{ and } y_1^2+\ldots\ldots+y_{r_1}^2$$

are transformable into each other modulo P^l, where $l=e'_2-e_1$ and $k=e_2-e_1$. Let the second of these be transformed into the first by

$$T:\ y_i\equiv\sum_{j=1}^{n} c_{ij}x_j \pmod{P^l}.$$

Employing partial derivatives with respect to x_i, we have

$$S:\left\{\begin{array}{l} x_1\equiv c_{11}y_1+\ldots\ldots+c_{r_11}y_{r_1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{r_1}\equiv c_{1r_1}y_1+\ldots\ldots+c_{r_1r_1}y_{r_1} \end{array}\right\} \pmod{P^l}.$$

* L. E. Dickson, The Madison Colloquium Lectures, 1913, p. 8.

The inverse of S gives y_i $(i=1, \ldots, r_1)$ congruent to linear homogeneous functions of $x_1, \ldots, x_{r_1}$ modulo P^l. Hence, if (6_1) and (7_1) are equivalent modulo P^λ, $P^k q \equiv 0 \pmod{P^l}$ for all integral values of $x_{r_1+1}, \ldots, x_n$. Since at least one of the coefficients in q is prime to P, this implies that $k \geq l$, *i. e.*, that $e_2 \geq e_2'$. Similarly, e_2' can not be less than e_2. Therefore, the choice of the principal minor of $|A_{ij}^{(1)}|$ that is prime to P is immaterial so far as the value of e_2 is concerned. Consider now the two forms

$$Q = x_1^2 + \ldots + x_{r_1}^2 + P^l q, \quad Q' = y_1^2 + \ldots + y_{r_1}^2 + P^l q'.$$

Let Q and Q' be equivalent modulo $P^{\lambda-e_1}$ and let

$$T:\ y_i \equiv \sum_{j=1}^{n} c_{ij} x_j \pmod{P^{\lambda-e_1}} \qquad (i=1, \ldots, n)$$

transform Q' into Q modulo $P^{\lambda-e_1}$, where $|c_{ij}| \not\equiv 0 \pmod{P}$. Then we have

$$S:\ \left\{\begin{array}{l} \dfrac{\partial Q}{\partial x_1} = 2x_1 \equiv 2c_{11}y_1 + \ldots + 2c_{r_1 1}y_{r_1} + P^l \dfrac{\partial q'}{\partial x_1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{\partial Q}{\partial x_{r_1}} = 2x_{r_1} \equiv 2c_{1r_1}y_1 + \ldots + 2c_{r_1 r_1}y_{r_1} + P^l \dfrac{\partial q'}{\partial x_{r_1}} \end{array}\right\} \pmod{P^{\lambda-e_1}},$$

which shows that $|c_{ij}|$ $(i, j=1, \ldots, r_1)$ is prime to P. The inverse of S gives $y_1, \ldots, y_{r_1}$ expressed as linear homogeneous functions of $x_1, \ldots, x_{r_1}$, $y_{r_1+1}, \ldots, y_n$. Moreover, the coefficients of $y_{r_1+1}, \ldots, y_n$ in these functions are divisible by P^l. Hence T is expressible in the form

$$y_i \equiv \sum_{j=1}^{r_1} c_{ij} x_j + P^l \sum_{j=r_1+1}^{n} c'_{ij} y_j \pmod{P^{\lambda-e_1}} \qquad (i=1, \ldots, r_1),$$

$$y_i \equiv \sum_{j=1}^{n} c_{ij} x_j \pmod{P^{\lambda-e_1}} \qquad (i=r_1+1, \ldots, n).$$

Subjecting Q' to this transformation, we have

$$\phi(x_1, \ldots, x_{r_1}) + P^l \theta(x_1, \ldots, x_n).$$

Since, by hypothesis, T transforms Q' into Q modulo $P^{\lambda-e_1}$, the terms in θ involving $x_1, \ldots, x_{r_1}$ must vanish modulo $P^{\lambda-e_1}$, and we have

$$\phi \equiv x_1^2 + \ldots + x_{r_1}^2 \pmod{P^{\lambda-e_1}}.$$

From this we see that the equivalence of (6_1) and (7_1) modulo P^λ implies the equivalence of $P^{e_2} q$ and $P^{e_2} q'$. Similarly, for the equivalence of (6_2) and (7_2). This shows that in the method of reduction of a quadratic form to the canonical form (8) or (9) modulo P^λ, it is immaterial whether we effect it by employing principal minors or by a seriatim process as was done by Jordan in a similar case.* This seriatim process is given in the argument leading up to (3).

* *Journal de Mathématiques*, Ser. 2, Vol. XVII (1872), pp. 368–375.

Repetitions of the arguments leading to (5) and (6) show that there exists a linear homogeneous transformation of determinant unity modulo P^λ under which, when $|P^{-e_1}a_{ij}|$ is divisible by P, the form (1) becomes

$$P^{e_1}[x_1^2+\ldots\ldots+x_{r_1-1}^2+\rho^{m_1}x_{r_1}^2]+P^{e_2}[x_{r_1+1}^2+\ldots\ldots+x_{r_1+r_2-1}^2+\rho^{m_2}x_{r_1+r_2}] \\ +\ldots\ldots+P^{e_k}[x_{r_1+r_2+\ldots\ldots+r_{k-1}+1}^2+\ldots\ldots+x_{r_1+\ldots\ldots+r_k-1}^2+\rho^{m_k}x_{r_1+\ldots\ldots+r_k}^2] \qquad (8)$$

if $r_1+r_2+\ldots\ldots+r_k<n$, or

$$P^{e_1}[x_1^2+\ldots\ldots+x_{r_1-1}^2+\rho^{m_1}x_{r_1}^2]+P^{e_2}[x_{r_1+1}^2+\ldots\ldots+x_{r_1+r_2-1}^2+\rho^{m_2}x_{r_1+r_2}^2] \\ +\ldots\ldots+P^{e_k}[x_{r_1+\ldots\ldots+r_{k-1}+1}^2+\ldots\ldots+x_{n-1}^2+P^{-\sum\limits_{i=1}^{k}r_ie_i}\rho^{-\sum\limits_{i=1}^{k-1}m_i}Dx_n^2] \qquad (9)$$

if $r_1+r_2+\ldots\ldots+r_k=n$, where $m_i=0$ or 1 [$i=1,\ldots\ldots,k$ in (8); $i=1,\ldots\ldots,k-1$ in (9)], $D=|a_{ij}|$ and r_s is the rank modulo P of the determinant

$$|A_{ij}^{(s)}| \quad (i,j=r_1+r_2+\ldots\ldots+r_{s-1}+1,\ldots\ldots,n).$$

$A_{ij}^{(s)}=P^{-e_s}B_{ij}^{(s-1)}$, where the $B_{ij}^{(s-1)}$ are found from the $A_{ij}^{(s-1)}$ in the same way that the $B_{ij}^{(1)}$ were found from $A_{ij}^{(1)}$ in (4). Therefore, when not all the a_{ij} are zero modulo P^λ, we have the canonical forms (3), (8), (9), and from the work above it is evident that two forms are equivalent modulo P^λ, if and only if they have the same canonical form. Since (3) is a special case of (9) with $k=1$, *i. e.*, $r_1=n$, we need consider only the canonical forms (8), (9) and the vanishing form where $a_{ij}\equiv 0 \pmod{P^\lambda}$ $(i,j=1,\ldots\ldots,n)$. Moreover, since two forms are equivalent modulo P^λ if and only if they have the same canonical form, we have the classes C_0 and $Ce_im_iR_k$, defined by a canonical form of type (8) with $n>R_k=r_1+\ldots\ldots+r_k$; Ce_im_jnD, defined by a canonical form of type (9) with $n=R_k$, for $i=1,\ldots\ldots,k$; $j=1,\ldots\ldots,k-1$; $e_i<\lambda$.

2. *Elementary Exponents and Ranks.*—Consider the symmetric determinant $D=|a_{ij}|$ $(i,j=1,\ldots\ldots,n)$, where the elements are integers. If P^{α_i} is the highest power of P that divides all the i-rowed minors of D, then α_i is unchanged when D is pre-multiplied or post-multiplied by an n-th order determinant whose elements are integers and whose value is unity.* Frobenius called the P^{e_i} $(e_i=\alpha_i-\alpha_{i-1},\ e_1=\alpha_1)$ the elementary invariants of D with respect to P. We shall call the e_i the elementary exponents of D with respect to P, or simply the elementary exponents of D. Note for later use that† e_i is a monotonically increasing function of i. Let P^{e_1} be the highest power of P that divides all the a_{ij} and consider the case when $e_1<\lambda$. Let the rank modulo P of

* Stephen Smith, *Phil. Trans.*, Vol. CLI, § 12, p. 311. Bachmann's *Zahlentheorie*, Vol. IV. Frobenius, *Berlin Sitzungsberichte* (1894), Pt. I, p. 31. These properties hold if D is not symmetric, but our interest lies in the theorem as given.

† Frobenius, *loc. cit.* Also seen from this section.

the determinant $|A_{ij}|=|P^{-e_1}a_{ij}|$ be r_1, $0<r_1<n$. Then there exists* a principal minor M_{r_1} of $|A_{ij}|$ of order r_1 that is prime to P. By multiplying D by properly chosen unit determinants we may assume that

$$|A_{ij}|=M_{r_1}\not\equiv 0 \pmod P \qquad (i, j=1, \ldots, r_1).$$

Consider the unit n-rowed determinant $d^{(1)}=|d^{(1)}_{st}|$ in which each element is zero except

$$d^{(1)}_{st}=1 \text{ when } s=t, \quad d^{(1)}_{r_1+1,t}\equiv -\frac{A_{1,r_1+1}\overline{A}_{1t}}{M_{r_1}} \pmod{P^\lambda} \qquad (t=1, \ldots, r_1),$$

where $\overline{A}_{ij}$ denotes the cofactor of A_{ij} in M_{r_1}. If we pre-multiply D by $d^{(1)}$ we have, upon denoting the resulting elements by $b^{(1)}_{ij}$,

$$b^{(1)}_{1,r_1+1}\equiv a_{1,r_1+1}-\frac{a_{1,r_1+1}}{M_{r_1}}\sum_{t=1}^{r_1} A_{1t}\overline{A}_{1t}\equiv 0 \pmod{P^\lambda},$$

$$b^{(1)}_{k,r_1+1}\equiv a_{k,r_1+1}-\frac{a_{1,r_1+1}}{M_{r_1}}\sum_{t=1}^{r_1} A_{kt}\overline{A}_{1t} \pmod{P^\lambda} \qquad (k\neq 1).$$

Hence

$$b^{(1)}_{k,r_1+1}\equiv a_{k,r_1+1} \pmod{P^\lambda} \qquad (1<k\leqq r_1),$$

$$b^{(1)}_{k,r_1+1}\equiv \frac{P^{e_1}}{M_{r_1}}[A_{k,r_1+1}M_{r_1}-A_{1,r_1+1}M^{(k)}_{r_1 1}] \qquad (k>r_1),$$

where $M^{(k)}_{r_1 1}$ denotes the determinant formed from M_{r_1} by replacing its first row or column by the corresponding elements of the k-th row or column of $|A_{ij}|$ $(i, j=1, \ldots, n)$. All other elements of D are unchanged. If we now pre-multiply this resulting determinant by $d^{(2)}=|d^{(2)}_{st}|$, where $s, t=1, \ldots, n$, and every element is zero except

$$d^{(2)}_{st}=1 \text{ when } s=t; \quad d^{(2)}_{r_1+1,t}\equiv -\frac{A_{2,r_1+1}\overline{A}_{2t}}{M_{r_1}} \qquad (t=1, \ldots, r_1);$$

we have

$$b^{(2)}_{1,r_1+1}\equiv b^{(2)}_{2,r_1+1}\equiv 0 \pmod{P^\lambda}, \quad b^{(2)}_{k,r_1+1}\equiv a_{k,r_1+1} \qquad (k=3, 4, \ldots, r_1),$$

$$b^{(2)}_{k,r_1+1}\equiv \frac{P^{e_1}}{M_{r_1}}[A_{k,r_1+1}M_{r_1}-A_{1,r_1+1}M^{(k)}_{r_1 1}-A_{2,r_1+1}M^{(k)}_{r_1 2}] \pmod P \qquad (k>r_1).$$

All other elements are again unchanged. Continuing this process, we ultimately obtain from D a determinant having its first r_1 elements in the (r_1+1)-th row zero modulo P^λ and the k-th element of this row $(k>r_1)$ is

$$P^{e_1}B^{(1)}_{k,r_1+1}\equiv \begin{vmatrix} A_{11} & \ldots A_{1r_1} & A_{1k} \\ \cdots & \cdots\cdots & \cdots \\ A_{r_1 1} & \ldots A_{r_1 r_1} & A_{r_1 k} \\ A_{r_1+1,1} & \ldots A_{r_1+1,r_1} & A_{r_1+1,k} \end{vmatrix} P^{e_1}M_{r_1}^{-1} \pmod{P^\lambda}.$$

* L. E. Dickson, *Annals of Mathematics*, Ser. 2, Vol. XV (1913), pp. 27, 28.

$B^{(1)}_{k,r_1+1}$ is divisible by $P(r_1<k\leqq n)$, since r_1 is the rank modulo P of the determinant $|A_{ij}|$ $(i, j=1, \ldots, n)$. All the elements of D in any other row are unchanged. Operating similarly upon the (r_1+2)-th, $\ldots$, n-th row of this determinant and then post-multiplying by a properly chosen unit determinant, we ultimately obtain from D

$$D^{(1)}\equiv\begin{vmatrix} A & O \\ O & B\end{vmatrix} \pmod{P^\lambda}, \quad A=\begin{pmatrix} a_{11} \ldots a_{1r_1} \\ \ldots\ldots\ldots \\ a_{r_11} \ldots a_{r_1r_1}\end{pmatrix}, \quad B=\begin{pmatrix} P^{e_1}B^{(1)}_{r_1+1r_1+1} \ldots P^{e_1}B^{(1)}_{r_1+1n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ P^{e_1}B^{(1)}_{nr_1+1} \ldots P^{e_1}B^{(1)}_{nn}\end{pmatrix},$$

where O is a matrix all of whose elements are zero.

Here $B^{(1)}_{ij}$ $(i, j=r_1+1, \ldots, n)$ is divisible by P. To determine the highest power of P that divides all the $P^{e_1}B^{(1)}_{ij}$, let $P^{e'_2}$ denote the highest power of P that divides all the (r_1+1)-rowed minors of D. Then $P^{e'_2}$ is the highest power of P that divides all the (r_1+1)-rowed minors of $D^{(1)}$. Then evidently $P^{e'_2-e_1r_1}$ is the highest power of P that divides all the $P^{e_1}B^{(1)}_{ij}$ of $D^{(1)}$. Let $e_2=e'_2-r_1e_1$. Then e_2 is the (r_1+1)-th elementary exponent of D. We shall call r_1 the first elementary rank of D with respect to P. Consider the case when $e_2<\lambda$. Let r_2 be the rank modulo P of the determinant

$$|P^{-(e_2-e_1)}B^{(1)}_{ij}| \qquad (i, j=r_1+1, \ldots n).$$

Since there is a principal minor of this determinant of order r_2 that is prime to P, we may operate on this determinant as we did on D above. Moreover, considering this determinant as it stands in $D^{(1)}$, we see that this operation would leave the first r_1 rows and columns of $D^{(1)}$ unaltered. Hence we would have

$$D^{(2)}\equiv\begin{vmatrix} A & O & O \\ O & B^{(1)} & O \\ O & O & C\end{vmatrix} \pmod{P^\lambda},$$

$$B^{(1)}=\begin{pmatrix} P^{e_1}B^{(1)}_{r_1+1,r_1+1} \ldots P^{e_1}B^{(1)}_{r_1+1,r_1+r_2} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ P^{e_1}B^{(1)}_{r_1+r_2,r_1+1} \ldots P^{e_1}B^{(1)}_{r_1+r_2,r_1+r_2}\end{pmatrix}, \quad C=\begin{pmatrix} P^{e_2}B^{(2)}_{r_1+r_2+1,r_1+r_2+1} \ldots P^{e_2}B^{(2)}_{r_1+r_2+1,n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ P^{e_2}B^{(2)}_{n,r_1+r_2+1} \ldots P^{e_2}B^{(2)}_{n,n}\end{pmatrix}.$$

Let $P^{e'_3}$ be the highest power of P that divides all the (r_1+r_2+1)-rowed minors of D. Then $e'_3-r_1e_1-r_2e_2=e_3$ is the highest power of P that divides all the $P^{e_2}B^{(2)}_{ij}$ of $D^{(2)}$. Evidently we may continue this process until all the new elements are zero modulo P^λ, or until $\Sigma r_i=n$. At any stage, say the t-th, e_t and r_t are determined by the preceding. In fact

$$e_t=e'_t-\sum_{i=1}^{t-1} r_ie_i,$$

where e'_t is the highest power of P that divides all the $(r_1+r_2+\ldots+r_{t-1}+1)$-rowed minors of D, and r_t is an integer such that $E=r_1e_1+\ldots+r_te_t$ is the

exponent of the highest power of P that divides all the $(r_1+r_2+\dots+r_t)$-rowed minors of D while all the $(r_1+\dots+r_t+1)$-rowed minors are divisible by a higher power of P than P^{E+e_t}. We shall refer to these r_t as the elementary ranks of D with respect to P.

Definition. The determinant D shall be said to be of rank R modulo P^λ if the R-th elementary exponent of D with respect to P is less than λ, while the $(R+1)$-th elementary exponent of D is $\geqq \lambda$.

From this definition it follows that the rank of D modulo P^λ is an invariant under multiplication of D by unit determinants and that under such multiplication D becomes

$$D^{(k)} \equiv \begin{vmatrix} A & 0 & 0 \dots 0 & 0 & 0 \dots \\ 0 & B^{(1)} & 0 \dots 0 & 0 & 0 \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \dots 0 & B^{(k)} & 0 \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} \pmod{P^\lambda},$$

$$B^{(k)} = \begin{pmatrix} P^{e_k} B^{(k)}_{R_k+1,\,R_k+1} \dots P^{e_k} B^{(k)}_{R_k+1,\,R_{k+1}} \\ \dots\dots\dots\dots \\ P^{e_k} B^{(k)}_{R_{k+1},\,R_k+1} \dots P^{e_k} B^{(k)}_{R_{k+1},\,R_{k+1}} \end{pmatrix},$$

where $R_i = r_1 + \dots + r_i$, and R_k is the rank of D modulo P^λ. The $e_1, e_2, \dots, e_k$ are the distinct elementary exponents of D that are less than λ arranged in the order of increasing magnitude. From these considerations it follows that if the rank modulo P^λ of a symmetric determinant D is $R > 0$, there exists a principal minor M_R of D of order R whose rank modulo P^λ is R, and such that the highest power of P that divides all the R-rowed minors of D is exactly the power of P that is contained in M_R, or, more precisely, whose elementary exponents are those of D that are less than λ and whose elementary ranks are precisely those of D. In fact, the M_{R_k} formed by taking the first R_k rows and columns of $D^{(k)}$ is congruent modulo P^λ to such a principal minor of D of order R_k. From the form of $D^{(k)}$ also follows the fact that the elementary exponent is a monotonically increasing function of the order of the minors.

3. *Criteria for the Classes $Ce_i m_i R_k$ and $Ce_i m_i nD$.*—It follows from §§ 1 and 2 that the determinant D of a quadratic form, the e_i, r_i of D and the rank $R_k = r_1 + \dots + r_k$ of D modulo P^λ are invariants. However, these invariants do not completely characterize the classes. We seek criteria for the determination of the m_i of (8) and (9) § 1. Consider the modular form q_n given by (1) and let not all the a_{ij} be zero modulo P^λ. Under transformations of the type used in § 1, q_n becomes

$$\sum_{s=1}^{k} P^{e_s} \sum_{i,\,j=R_{s-1}+1}^{R_s} A^{(s)}_{ij} x_i x_j, \quad R_s = \sum_{i=1}^{s} r_i, \tag{10}$$

where $e_1 < e_2 < \ldots . < e_k$ are the distinct elementary exponents of D wit respect to P that are less than λ, and the r_i are the elementary ranks of 1 with respect to P. The determinants $|A_{ij}^{(i)}|$ are prime to P, and R_k is the ran modulo P^λ of D.

Consider first the case $R_k < n$. Let $E_i = r_1 e_1 + \ldots . + r_i e_i$ and denote th minor $|P^{e_1} A_{ij}^{(1)}|$ of the determinant of (10) by M_{R_1}. Then q_n belongs to th class $Ce_i m_i R_k$ with $m_1 = 0$ or 1, according as $(P^{-E_1} M_{R_1})^{\frac{\mu}{2}} \equiv 1$ or -1 (mod P^λ) where $\mu = \phi(P^\lambda) = P^{\lambda-1}(P-1)$. By §§ 1, 2, M_{R_1} is congruent modulo P^λ to a R_1-rowed principal minor of D such that $P^{-E_1} M_{R_1}$ is prime to P. Let now M_R denote any such principal minor of D. Then, since q_n has one and but one o the canonical forms, we conclude that q_n belongs to the class $Ce_i m_i R_k$ wit $m_1 = 0$ or 1 according as $(P^{-E_1} M_{R_1})^{\frac{\mu}{2}} \equiv 1$ or -1 (mod P^λ). Consider next th minor

$$M_{R_2} \equiv \begin{vmatrix} P^{e_1} A_{11}^{(1)} & \ldots . P^{e_1} A_{1,R_1}^{(1)} & 0 \ldots \ldots \ldots \ldots . 0 \\ \ldots \ldots \ldots \ldots & & \ldots \ldots \ldots \ldots \\ P^{e_1} A_{R_1,1}^{(1)} & \ldots . P^{e_1} A_{R_1 R_1}^{(1)} & 0 \ldots \ldots \ldots \ldots . 0 \\ 0 \ldots \ldots \ldots & 0 \ldots . & P^{e_2} A_{R_1+1,R_1+1}^{(2)} \ldots . P^{e_2} A_{R_1+1,R_2}^{(2)} \\ \ldots \ldots \ldots \ldots & & \ldots \ldots \ldots \ldots \\ 0 \ldots \ldots \ldots & 0 \ldots . & P^{e_2} A_{R_2,R_1+1}^{(2)} \ldots . P^{e_2} A_{R_2 R_2}^{(2)} \end{vmatrix}.$$

M_{R_2} is an R_2-rowed principal minor of the determinant of (10) and is con gruent modulo P^λ to an R_2-rowed principal minor of D. Moreover, by §§ 1, 2 q_n belongs to the class $Ce_i m_i R_k$ with $m_2 = 0$ or 1 according as

$$(|A_{ij}^{(2)}|)^{\frac{\mu}{2}} = \left(\frac{P^{-E_2} M_{R_2}}{|A_{ij}^{(1)}|}\right)^{\frac{\mu}{2}} \equiv (|A_{ij}^{(1)}|)^{\frac{\mu}{2}} (P^{-E_2} M_{R_2})^{\frac{\mu}{2}} \equiv 1 \text{ or } -1 \pmod{P^\lambda}.$$

Let now M_{R_2} denote any R_2-rowed principal minor of D having an R_1-rowe principal minor M_{R_1} such that $P^{-E_2} M_{R_2}$ and $P^{-E_1} M_{R_1}$ are prime to P. Then, i view of §§ 1, 2, and the fact that q_n has one and but one of the canonical forms q_n belongs to the class $Ce_i m_i R_k$ with $m_2 = 0$ or 1 according as

$$(P^{-E_1} M_{R_1})^{\frac{\mu}{2}} (P^{-E_2} M_{R_2})^{\frac{\mu}{2}} \equiv 1 \text{ or } -1 \pmod{P^\lambda}.$$

In general let M_{R_s} be any R_s-rowed principal minor of D having an R_{s-1}-rowe principal minor $M_{R_{s-1}}$, which in turn has an R_{s-2}-rowed principal minor $M_{R_{s-}}$ and so on to M_{R_2} which has an R_1-rowed principal minor M_{R_1} such tha $P^{-E_i} M_{R_i}$ $(i = 1, \ldots, s)$ is prime to P. Then by a transformation of deter minant unity which merely permutes the x_i of (1) we have the principal minor M_{R_1} occupying the first R_1 rows and columns of D, etc. Then from the forn of $D^{(k)}$ of § 2 we have that q_n belongs to the class $Ce_i m_i R_k$ with $m_s = 0$ or 1 according as

$$\prod_{j=1}^{s} (P^{-E_j} M_{R_j})^{\frac{\mu}{2}} \equiv 1 \text{ or } -1 \pmod{P^\lambda}. \qquad (11)$$

For the case $R_k=n$, D together with (11) for $s=1, \ldots, k-1$ furnish criteria. At the first consideration one would probably be surprised that D may be zero modulo P^λ and still q_n belong to class $Ce_i m_j nD$. In fact D may be divisible by a much higher power of P than P^λ and still q_n belong to class $Ce_i m_j nD$, since it is not the power of P that divides the minors of D that determine the number of terms in the canonical form, but the elementary exponents of D. This phenomenon is what gave rise to the definition of the rank of D modulo P^λ of § 2. The rank of D modulo P^λ shall be called the rank of the form q_n modulo P^λ. Two forms q_n and q'_n of rank n are equivalent if and only if

$$e_i=e'_i,\ r_i=r'_i,\ m_j=m'_j \qquad (i=1, \ldots, k;\ j=1, \ldots, k-1)$$

and $P^{-E_k+e_k}D \equiv P^{-E'_k+e'_k}D' \pmod{P^\lambda}$ as is seen by §§ 1, 2.

4. *Invariantive Criteria for the Classes.*—Consider the modular quadratic form (1), and let $e_1<e_2<\ldots<e_k$ be the distinct elementary exponents of D with respect to P. Let $\overline{D}$, $\bar{e}_i$, $\bar{r}_i$ and $\overline{m}_i$ be a consistent set of values that define a class $C\bar{e}_i\overline{m}_i\overline{R}_k$ or $C\bar{e}_i\overline{m}_j n\overline{D}$ according as the rank $\overline{R}_k$ modulo P^λ of $\overline{D}$ is less than or equal to n. For abbreviation set

$$\begin{aligned}
&\mathfrak{R}_i=\sum_{j=1}^{i-1} r_j+\bar{r}_i, && \mathfrak{E}_i=\sum_{j=1}^{i-1} r_j e_j+\bar{r}_i e_i && (i=1, \ldots, k),\\
&\overline{R}_i=\sum_{j=1}^{i} \bar{r}_j, && \overline{E}_i=\sum_{j=1}^{i} \bar{r}_j \bar{e}_j && (i=1, \ldots, k),\\
&R'_i=\sum_{j=1}^{i-1} r_j+r'_i, && E'_i=\sum_{j=1}^{i-1} r_j e_j+r'_i e_i && (i=1, \ldots, k;\ \bar{r}_i<r'_i\leqq n-R_{i-1}),\\
&R_i=\sum_{j=1}^{i} r_j, && E_i=\sum_{j=1}^{i} r_j e_j && (i=1, \ldots, h).
\end{aligned}$$

Thus R_k and $\overline{R}_k$ are the ranks modulo P^λ of D and $\overline{D}$, respectively.

Consider the function *

$$\begin{aligned}
A_{\overline{R}_1}(D) \equiv (P^{e_1-\bar{e}_1})^\mu \{ (P^{-\mathfrak{E}_1}M^{(1)}_{\mathfrak{R}_1})^{\frac{\mu}{2}} + (P^{-\mathfrak{E}_1}M^{(2)}_{\mathfrak{R}_1})^{\frac{\mu}{2}}[1-(P^{-\mathfrak{E}_1}M^{(1)}_{\mathfrak{R}_1})^\mu] \\
+\ldots+(P^{-\mathfrak{E}_1}M^{(r)}_{\mathfrak{R}_1})^{\frac{\mu}{2}}\prod_{j=1}^{r-1}[1-(P^{-\mathfrak{E}_1}M^{(j)}_{\mathfrak{R}_1})^\mu]\}\prod_{i}[1-(P^{-E'_1}M^{(i)}_{R'_1})^\mu] \pmod{P^\lambda}, \quad (12)
\end{aligned}$$

where $M^{(j)}_{\mathfrak{R}_1}$ ranges over the principal minors of D of order $\mathfrak{R}_1=\overline{R}_1$, and $M^{(i)}_{R'_1}$ ranges over the principal minors of D of order $R'_1>\mathfrak{R}_1$. Since P^{e_1} divides all the a_{ij} of D, it is evident that there is no term of $A_{\overline{R}_1}(D)$ that has as factor P with a negative exponent. $A_{\overline{R}_1}(D)$ is zero modulo P^λ, unless $e_1=\bar{e}_1$ and unless at least one of the $P^{-\mathfrak{E}_1}M_{\mathfrak{R}_1}$ is prime to P (since $P\geqq 3$). This implies that $A_{\overline{R}_1}(D)\equiv 0 \pmod{P^\lambda}$ unless the rank modulo P of the determinant $|P^{-e_1}a_{ij}|$ is $\bar{r}_1$. In fact, if $r_1<\bar{r}_1$, none of the $P^{-\mathfrak{E}_1}M_{\mathfrak{R}_1}$ are prime to P, while if $r_1>\bar{r}_1$, at least one

* In case $k=1$ and $\lambda>1$, $A_{\overline{R}_1}(D)$ is to have also the final factor of (15).

of the $P^{-E'_1}M_{R'_1}$ is prime to P and the final factor is zero by Fermat's theorem. Thus the value of $A_{\overline{R}_1}(D)$ is zero modulo P^λ, unless q_n belongs to a class $Ce_im_iR_k$ or Ce_im_jnD with $e_1=\overline{e}_1$ and $r_1=\overline{r}_1$, and by (11), § 3, its value modulo P^λ is 1 or -1 according as $m_1=0$ or 1.

Consider next the function

$$A_{\overline{R}_2}(D) \equiv (P^{e_2-\overline{e}_2})^\mu \{A_{\overline{R}_1}(M^{(1)}_{\mathfrak{R}_2})(P^{-\mathfrak{E}_2}M^{(1)}_{\mathfrak{R}_2})^{\frac{\mu}{2}} + A_{\overline{R}_1}(M^{(2)}_{\mathfrak{R}_2})(P^{-\mathfrak{E}_2}M^{(2)}_{\mathfrak{R}_2})^{\frac{\mu}{2}}[1-(P^{-\mathfrak{E}_2}M^{(1)}_{\mathfrak{R}_2})^\mu]$$
$$+\ldots + A_{\overline{R}_1}(M^{(s)}_{\mathfrak{R}_2})(P^{-\mathfrak{E}_2}M^{(s)}_{\mathfrak{R}_2})^{\frac{\mu}{2}}\prod_{j=1}^{s-1}[1-(P^{-\mathfrak{E}_2}M^{(j)}_{\mathfrak{R}_2})^\mu]\}\prod_i[1-P^{-E'_2}M^{(i)}_{R'_2})^\mu] \pmod{P^\lambda}, \quad (13)$$

where $M^{(j)}_{\mathfrak{R}_2}$ ranges over the principal minors of D of order $\mathfrak{R}_2$, and $M^{(i)}_{R'_2}$ ranges over the principal minors of D of orders $R'_2>\mathfrak{R}_2$. That the principal minors are divisible by the power of P indicated in (13) follows from the fact that the elementary exponents of D increase monotonically with the order of the minors to which they correspond. In fact, by hypothesis, every minor of order R_1 of D is divisible by $P^{r_1e_1}$ while every minor of order R_1+1 is divisible by $P^{r_1e_1+e_2}$. Let ε be the (R_1+2)-th elementary exponent of D. Then every minor of D of order R_1+2 is divisible by $P^{r_1e_1+e_2+\varepsilon}$. From the fact that $\varepsilon \geqq e_2$ we see that every minor of D of order R_1+2 is divisible by $P^{r_1e_1+2e_2}$. Continuing this we have that every minor of D of order $R_1+\overline{r}_2=\mathfrak{R}_2$ is divisible by $P^{r_1e+r_2e_2}=P^{\mathfrak{E}_2}$. If none of the $P^{-\mathfrak{E}_2}M_{\mathfrak{R}_2}$ are prime to P, $A_{\overline{R}_2}(D)\equiv 0 \pmod{P^\lambda}$. Its value is also zero modulo P^λ unless $r_1=\overline{r}_1$, $e_1=\overline{e}_1$ and $e_2=\overline{e}_2$ from the fact that each term carries an $A_{\overline{R}_1}$ function as factor and from the first factor. Thus $A_{\overline{R}_2}(D)$ is zero modulo P^λ unless at least one of the $M^{(j)}_{\mathfrak{R}_2}$ is such that $P^{-\mathfrak{E}_2}M_{\mathfrak{R}_2}$ is prime to P and at the same time contains a principal minor $M_{\mathfrak{R}_1}$ such that $P^{-\mathfrak{E}_1}M_{\mathfrak{R}_1}$ is prime to P. Let all these conditions be satisfied and take this $P^{-\mathfrak{E}_1}M_{\mathfrak{R}_1}=P^{-\overline{E}_1}M_{\overline{R}_1}$ as the M_{R_1} of §§ 1 and 2. Then we have that q_n is equivalent modulo P^λ to a form whose determinant is $D^{(1)}$ of § 2. If $r_2<\overline{r}_2$, all the $\mathfrak{R}_2$-rowed minors of $D^{(1)}$ are divisible by a higher power of P than $P^{\mathfrak{E}_2}$, and hence none of the $P^{-\mathfrak{E}_2}M_{\mathfrak{R}_2}$ of D are prime to P. Therefore, $r_2\geqq\overline{r}_2$. If $r_2>\overline{r}_2$, at least one of the R'_2-rowed principal minors of $D^{(1)}$ that contains $M_{\mathfrak{R}_1}$ as its leading R_1-rowed principal minor is such that $P^{-E'_2}M_{R'_2}$ is prime to P, and $A_{\overline{R}_2}(D)$ is zero modulo P^λ from its final factor by Fermat's Theorem. Hence $A_{\overline{R}_2}(D)\equiv 0 \pmod{P^\lambda}$ unless q_n belongs to a class $Ce_im_iR_k$ or Ce_im_jnD with $e_1=\overline{e}_1$, $e_2=\overline{e}_2$, $r_1=\overline{r}_1$, $r_2=\overline{r}_2$, and by (11) its value is 1 or -1 modulo P^λ according as $m_2=0$ or 1. Similarly, we have that the function $(i=1,\ldots,k-1)$

$$A_{\overline{R}_i}(D)\equiv(P^{e_i-\overline{e}_i})^\mu\{\prod_{j=1}^{i-1}A_{\overline{R}_j}(M^{(1)}_{\mathfrak{R}_i})(P^{-\mathfrak{E}_i}M^{(1)}_{\mathfrak{R}_i})^{\frac{\mu}{2}}$$
$$+\prod_{j=1}^{i-1}A_{\overline{R}_j}(M^{(2)}_{\mathfrak{R}_i})(P^{-\mathfrak{E}_k}M^{(2)}_{\mathfrak{R}_i})^{\frac{\mu}{2}}[1-(P^{-\mathfrak{E}_i}M^{(1)}_{\mathfrak{R}_i})^\mu]$$
$$+\ldots+\prod_{j=1}^{i-1}A_{\overline{R}_j}(M^{(t)}_{\mathfrak{R}_i})(P^{-\mathfrak{E}_i}M^{(t)}_{\mathfrak{R}_i})^{\frac{\mu}{2}}\prod_{v=1}^{t-1}[1-(P^{-\mathfrak{E}_i}M^{(v)}_{\mathfrak{R}_i})^\mu]\}\prod_w[1-(P^{-E'_i}M^{(w)}_{R'_i})^\mu]$$
$$\pmod{P^\lambda}, \quad (14)$$

where $M^{(v)}_{\mathfrak{R}_i}$ ranges over the principal minors of D of order $\mathfrak{R}_i$, and $M^{(w)}_{R'_i}$ ranges over the principal minors of D of order $R'_i > \mathfrak{R}_i$, has the value zero unless q_n belongs to a class $Ce_i m_i R_k$ or to a class $Ce_i m_j nD$ with $e_j = \bar{e}_j$, $r_j = \bar{r}_j$, and that for such forms its value is 1 or -1 modulo P^λ according as $m_i = 0$, or 1. When $\bar{R}_k < n$, we have

$$\begin{aligned} A_{\bar{R}_k}(D) \equiv (P^{e_k - \bar{e}_k})^\mu \{ \prod_{i=1}^{k-1} A_{\bar{R}_i}(M^{(1)}_{\mathfrak{R}_k})\,(P^{-\mathfrak{E}_k} M^{(1)}_{\mathfrak{R}_k})^{\frac{\mu}{2}} \\ + \prod_{i=1}^{k-1} A_{\bar{R}_i}(M^{(2)}_{\mathfrak{R}_k})\,(P^{-\mathfrak{E}_k} M^{(2)}_{\mathfrak{R}_k})^{\frac{\mu}{2}} [1 - (P^{-\mathfrak{E}_k} M^{(1)}_{\mathfrak{R}_k})^\mu] \\ + \dots + \prod_{i=1}^{k-1} A_{\bar{R}_i}(M^{(t)}_{\mathfrak{R}_k})\,(P^{-\mathfrak{E}_k} M^{(t)}_{\mathfrak{R}_k})^{\frac{\mu}{2}} \prod_{j=1}^{t-1} [1 - (P^{-\mathfrak{E}_k} M^{(j)}_{\mathfrak{R}_k})^\mu] \} \\ \prod_v [1 - (P^{-E'_k} M^{(v)}_{R'_k})^\mu] \prod_w [1 - (P^{-(E_k + e_{k+1})} d^{(w)}_{R_k+1})^\mu] \pmod{P^\lambda}, \quad (15) \end{aligned}$$

where $M^{(j)}_{\mathfrak{R}_k}$ ranges over the principal minors of D of order $\mathfrak{R}_k$, $M^{(v)}_{R'_k}$ ranges over those of order $R'_k > \mathfrak{R}_k$, $d^{(w)}_{R_k+1}$ ranges over the minors of D of order $R_k + 1$, and e_{k+1} is the (R_k+1)-th elementary exponent of D. In case $e_{k+1} \geqq \lambda$, define $P^{-(E_k + e_{k+1})} d^{(w)}_{R_k+1}$ to be zero modulo P^λ. Then $A_{\bar{R}_k}(D)$ has the value zero unless $e_i = \bar{e}_i$, $r_i = \bar{r}_i$ $(i = 1, \dots, k)$ and $e_{k+1} \geqq \lambda$, whence $R_h = R_k = \bar{R}_k$. Hence $A_{\bar{R}_k}(D) \equiv 0 \pmod{P^\lambda}$ unless q_n belongs to a class $Ce_i m_i R_k$ with $e_i = \bar{e}_i$ and $r_i = \bar{r}_i$ $(i = 1, \dots, k)$, and for such forms its value is 1 or -1 according as $m_k = 0$ or 1. When $\bar{R}_k = n$ we have

$$A_n(D) \equiv (P^{e_k - \bar{e}_k})^\mu (P^{-E_k} D)^\mu \{ 1 - [(P^{-E_k + e_k} D - P^{-\bar{E}_k + \bar{e}_k} \bar{D}) P^{-q}]^\mu \} \prod_{i=1}^{k-1} A_{\bar{R}_i}(D) \pmod{P^\lambda}, \quad (16)$$

where P^q is the highest power of P that divides $P^{-E_k + e_k} D - P^{-\bar{E}_k + \bar{e}_k} \bar{D}$. In case $q \geqq \lambda$, define $P^{-q}(P^{-E_k + e_k} D - P^{-\bar{E}_k + \bar{e}_k} \bar{D})$ to be zero modulo P^λ. Then the value of $A_n(D)$ is zero modulo P^λ unless $e_i = \bar{e}_i$, $r_i = \bar{r}_i$ $(i = 1, \dots, k)$ and $P^{-E_k + e_k} D \equiv P^{-\bar{E}_k + \bar{e}_k} \bar{D} \pmod{P^\lambda}$. The function is also zero unless $P^{-E_k} D$ is prime to P, *i. e.*, unless the rank modulo P^λ of D is R_k. Hence $A_n(D)$ is zero modulo P^λ unless q_n belongs to a class $Ce_i m_j nD$ with $e_i = \bar{e}_i$, $r_i = \bar{r}_i$ $(i = 1, \dots, k)$ and its value is 1 modulo P^λ for all such forms.

Finally, define $P^{-e_1} a_{ij}$ to be zero modulo P^λ when $e_1 \geqq \lambda$. Then

$$I \equiv \prod [1 - (P^{-e_1} a_{ij})^\mu] \pmod{P^\lambda} \qquad (i, j = 1, \dots, n;\ i \leqq j) \qquad (17)$$

has the value zero modulo P^λ unless q_n belongs to the class C_0 and the value 1 for such a form. The functions $A_{\bar{R}_i}(D)$, $A_n(D)$ and I are modular invariants that completely characterize the classes of modular quadratic forms modulo P^λ.

5. *Characteristic Modular Invariants.*—An invariant ϕ that has the value 1 for all forms of one class, and the value zero for forms of any other

class is called a characteristic invariant. In case such a ϕ is a modular invariant it is called a characteristic modular invariant. For example, I is a characteristic invariant for the class C_0.

Let $\bar{e}_i$, $\bar{r}_i$, $\bar{m}_i$ $(i=1, \ldots, k)$ be a set of consistent values that define the class $C\bar{e}_i\bar{m}_i\bar{R}_k$. Among the integers $1, \ldots, k$ a certain number s_1 of them will correspond to m's that are zero. The remaining s_2 of them will correspond to m's that are unity. Call the first set σ_1, and the second, σ_2, and consider the function

$$A_{\bar{R}_k\sigma_1\sigma_2} \equiv \frac{1}{2^{s_1+s_2}} \prod_i [(A_{\bar{R}_i}(D))^2 + A_{\bar{R}_i}(D)] \prod_j [(A_{\bar{R}_j}(D))^2 - A_{\bar{R}_j}(D)] \pmod{P^\lambda}, \quad (18)$$

where i ranges over σ_1, and j ranges over σ_2. This function has the value 1 * for all forms of class $C\bar{e}_i\bar{m}_i\bar{R}_k$ and the value zero for all other forms. Hence $A_{\bar{R}_k\sigma_1\sigma_2}$ is a characteristic modular invariant of q_n.

Next let $\bar{e}_i$, $\bar{r}_i$ $(i=1, \ldots, k)$, $\bar{m}_j$ $(j=1, \ldots, k-1)$ and $\bar{D}$ be a consistent set of values of these quantities that define the class $C\bar{e}_i\bar{m}_j n\bar{D}$; thus $\bar{R}_k=n$. Separate the integers $1, \ldots, k-1$ into sets σ_1 and σ_2 composed of s_1 and s_2 elements, respectively, such that those of σ_1 correspond to the m's that are zero, while those of σ_2 correspond to the m's that are unity. Then the function

$$A_{n\sigma_1\sigma_2} \equiv \frac{1}{2^{s_1+s_2}} \prod_i [(A_{\bar{R}_i}(D))^2 + A_{\bar{R}_i}(D)] \prod_j [(A_{\bar{R}_j}(D))^2 - A_{\bar{R}_j}(D)] A_n(D) \pmod{P}, \quad (19)$$

where i ranges over σ_1 and j ranges over σ_2, has the value 1 for all forms of class $C\bar{e}_i\bar{m}_j n\bar{D}$ and the value zero for all other forms.

Hence the characteristic invariants $A_{\bar{R}_k\sigma_1\sigma_2}$, $A_{n\sigma_1\sigma_2}$, and I completely characterize the classes of modular quadratic forms modulo P^λ.

For the case $\lambda=1$, $A_{\bar{R}_0}(D)$ and $A_{\bar{R}_k\sigma_1\sigma_2}$ become, respectively, the invariants A_r and $I_{r,\pm 1}$ given by Professor Dickson in his Madison Colloquium lectures, pp. 11, 13. The function $A_{n\sigma_1\sigma_2}$ is for this case a characteristic invariant of his class C_{nD}, while I in this case is replaced by the simple form $I=\Pi(1-a_{ij}^{\mu})$, as given *loc. cit.*, p. 11. These are the only ones that enter in this case, since the only set of integers e_i, m_i, r_i are 0, r_1, m_1. Thus, for this case the invariants given above are all polynomial modular invariants. This may also happen for the case $\lambda > 1$, but would obviously be a very special case.

6. *Number of Linearly Independent Modular Invariants of q_n.*—For convenience denote the classes of modular quadratic forms modulo P^λ by C_0, C_1, C_2, $\ldots$ and the corresponding characteristic invariants by

* In case either σ_1 or σ_2 is an empty set, the corresponding product factors of (18) is defined to be unity. Similarly in (19).

$I_0, I_1, I_2, \ldots$. Since any modular invariant I of q_n takes certain values $v_0, v_1, v_2, \ldots$ for the respective classes $C_0, C_1, C_2, \ldots$ we have

$$I \equiv v_0 I_0 + v_1 I_1 + v_2 I_2 + \ldots \pmod{P^\lambda}.$$

Hence, any modular invariant is congruent to a linear homogeneous function of the characteristic invariants. Moreover, the number of linearly independent modular invariants of the quadratic form q_n modulo P^λ equals the number of classes.

II. Polynomial Modular Invariants of a Binary Quadratic Form.

7. *Invariants for a Prime Power Modulus* P^λ.—Consider the form

$$f = a_0 x^2 + a_1 xy + a_2 y^2, \tag{1}$$

where the a_i are integers taken modulo P^λ. The transformations *

$$\begin{aligned} R:\ & x = x' + Ty', && y = y', \\ S:\ & x = x', && y = ky', \quad (k \text{ prime to } P) \\ Q:\ & x = y', && y = -x', \end{aligned}$$

generate the group of all linear homogeneous transformations whose determinant is prime to P and coefficients are integers modulo P^λ. Transformations R, S, Q give the respective replacements

$$a_0' = a_0, \quad a_1' = a_1 + 2a_0 T, \quad a_2' = a_2 + a_1 T + a_0 T^2. \tag{2}$$

$$a_0' = a_0, \quad a_1' = ka_1, \quad a_2' = k^2 a_2. \tag{3}$$

$$a_0' = a_2, \quad a_1' = -a_1, \quad a_2' = a_0. \tag{4}$$

Theorem 1. *The form f has the absolute invariant modulo P^λ*

$$H = \prod_{i=0}^{2} (a_i^\mu - 1) \qquad \mu = \phi(P^\lambda) = P^{\lambda-1}(P-1).$$

Evidently replacements (3) and (4) leave H unaltered. It remains to show that H is unaltered by replacement (2). This is evident if a_0 is prime to P, since then $a_0^\mu - 1 \equiv 0 \pmod{P^\lambda}$. Hence let a_0 be divisible by P. Then a_1' is of the form $a_1 + Pq$ and, by induction on λ,

$$(a_1 + Pq)^{P^{\lambda-1}} \equiv a_1^{P^{\lambda-1}} \pmod{P^\lambda},$$

whence $a_1'^\mu \equiv a_1^\mu$. Hence $H' \equiv A(a_2'^\mu - 1)$, where

$$A = (a_0^\mu - 1)(a_1^\mu - 1).$$

Thus $H' \equiv H$ if either a_0 or a_1 is prime to P. In the contrary case, $a_2' = a_2 + Pq$ and $a_2'^\mu \equiv a_2^\mu$ as before, whence $H' \equiv H$ in all cases.

* Cf. C. Jordan, "Traité des Substitutions," p. 93.

The same method leads to

THEOREM 2. *The form f has the absolute invariants modulo 2^λ*

$$I=a_1^\mu=a_1^{2^{\lambda-1}},\quad Q=a_0^\mu a_1^\mu a_2^\mu.$$

8. *Fundamental System of Invariants Modulo 4.*

THEOREM 3. *Necessary and sufficient conditions that*

$$\alpha+\beta T+\gamma T^2+\delta T^3\equiv 0 \pmod 4$$

for all integral values of T are

$$\alpha\equiv 0,\quad 2\beta\equiv 2\gamma\equiv\beta+\gamma+\delta\equiv 0 \pmod 4.$$

The general invariants of § 7 become

$$H=\prod_{i=0}^{2}(a_i^2-1),\quad I=a_1^2,\quad Q=a_0^2a_1^2a_2^2.$$

A useful combination of these invariants is

$$S=P-Q-I+1=a_0^2+a_2^2-a_0^2a_2^2-a_0^2a_1^2-a_1^2a_2^2.$$

THEOREM 4. *The form f has the absolute invariants* *

$$\begin{aligned}
K&\equiv a_1+a_1^3+2a_0a_2,\\
J&\equiv 2a_0+2a_2+a_0a_1+a_0a_1^2+a_1a_2+a_1a_2^3+a_0a_1a_2+a_0a_1^3a_2,\\
E&\equiv a_0a_2+a_0^2a_2^2+a_0a_2(a_1+a_1^2+a_1^3)\\
&\equiv a_0^3a_2+a_0a_2^2+a_0^2a_2^2-a_0^3a_2^2+a_0a_1a_2^2+a_0^3a_1^2a_2+a_0^3a_1^3a_2^2 \pmod 4.
\end{aligned}$$

The following identical congruences are of use in simplifying the discussion given below for the general invariant.

$$U\equiv(a_1+a_1^2)(a_2+a_2^2)\equiv 0,\quad V\equiv(a_0+a_0^2)(a_1+a_1^2)\equiv 0,\quad L\equiv(a_0+a_0^2)(a_2+a_2^2)\equiv 0,$$

$$Q_1=3Q\equiv a_0^3a_1^2a_2^2+a_0^3a_1^3a_2^2+a_0^2a_1^3a_2^2,\quad Q_2=3Q\equiv a_0^2a_1^2a_2+a_0^3a_1^2a_2+a_0^3a_1^2a_2^2,$$

$$M\equiv a_0a_1a_2^2+a_0^3a_1a_2^2+a_0a_1^2a_2+a_0^3a_1^2a_2\equiv 4Q\equiv 0,$$

$$N\equiv a_0a_1a_2^2+a_0^3a_1a_2^2+a_0a_1^3a_2+a_0^3a_1^3a_2\equiv 4Q\equiv 0.$$

In view of the identical congruence $x^4\equiv x^2 \pmod 4$, we may assume that the exponent of each a_i in an invariant of f does not exceed 3. Hence, we may denote any invariant ϕ by

$$\phi\equiv\sum_{i,j=0}^{3}A_{ij}a_1^ia_2^j,\quad A_{ij}=\alpha_{ij}+\beta_{ij}a_0+\gamma_{ij}a_0^2+\delta_{ij}a_0^3, \tag{5}$$

where α_{ij}, etc., are constants. The difference

$$\phi-(A_{11}+A_{13}a_2+A_{31}a_1+A_{33}a_1a_2)U$$

has

$$A_{11}\equiv A_{13}\equiv A_{31}\equiv A_{33}\equiv 0.$$

* K, J and E were discovered by carrying through for special cases the discussion made below for finding the general invariant; they were verified to be invariant with respect to the generators R, S, Q.

Hence, we may assume that in the initial ϕ these quantities are zero modulo 4. Similarly, by considering in turn

$$\phi-[\beta_{01}+(\gamma_{01}-\beta_{01})a_0]L-(\delta_{01}-\gamma_{01}+\beta_{01})E,\quad \phi-\alpha_{10}K-[\beta_{10}+(\gamma_{10}-\beta_{10})a_0]V,$$
$$\phi-[\beta_{30}+(\gamma_{30}-\beta_{30})a_0]a_1V,\qquad \phi-[\beta_{03}+(\gamma_{03}-\beta_{03})a_0]a_2L,$$

we may assume that

$$A_{01}\equiv\alpha_{01},\quad A_{10}\equiv\delta_{10}a_0^3,\quad A_{30}\equiv\alpha_{30}+\delta_{30}a_0^3,\quad A_{03}\equiv\alpha_{03}+\delta_{03}a_0^3.$$

Then by employing in turn

$$\phi-\beta_{21}M-\gamma_{21}Q_2,\quad \phi-\beta_{32}N-\gamma_{32}Q_1,\quad \phi-\alpha_{12}J-\gamma_{12}Q_2,\quad \phi-\alpha_{20}I-\gamma_{22}Q+\alpha_{22}S,$$

we may take

$$A_{21}\equiv\alpha_{21}+\delta_{21}a_0^3,\quad A_{32}\equiv\alpha_{32}+\delta_{32}a_0^3,\quad A_{12}\equiv\beta_{12}a_0+\delta_{12}a_0^3,\quad \alpha_{20}=\alpha_{22}=\gamma_{22}=0.$$

At no step have any of the quantities previously taken to be zero been disturbed.

Subjecting the simplified ϕ to replacement (2) we get

$$\phi'\equiv\phi+\beta T+\gamma T^2+\delta T^3,$$

where

$$\begin{aligned}\beta\equiv{}&2A_{10}a_0+2A_{12}a_0+2A_{02}a_0a_2+2A_{12}a_0a_2+(A_{01}+2A_{02}a_0+2A_{22}a_0+2A_{30}a_0\\&+2A_{32}a_0+2A_{02}a_2+2A_{03}a_0a_2+2A_{12}a_2+2A_{12}a_0a_2+2A_{22}a_2+2A_{22}a_0a_2\\&+2A_{23}a_0a_2+2A_{32}a_2+3A_{03}a_2^2)a_1+(A_{21}+3A_{23}a_2^2)a_1^3,\\ \gamma\equiv{}&A_{01}a_0+A_{02}a_0^2+A_{03}a_0^3+3A_{03}a_0a_2^2+3A_{03}a_0^2a_2+A_{12}a_0^2a_1+(A_{02}+3A_{03}a_2+A_{21}a_0\\&+3A_{03}a_0+A_{22}+A_{23}a_0^2+3A_{23}a_0a_2^2+3A_{23}a_2+3A_{23}a_0^2a_2+3A_{23}a_0+A_{23}a_0^3)a_1^2\\&+(A_{12}+A_{32}+A_{32}a_0^2)a_1^3,\\ \delta\equiv{}&3A_{03}a_0^2a_1+(A_{03}+A_{23}+3A_{23}a_0^2)a_1^3.\end{aligned}$$

By Theorem 3, necessary and sufficient conditions that $\phi'-\phi\equiv0$ for all integral values of T are $2\beta\equiv2\gamma\equiv\beta+\gamma+\delta\equiv0$. From the form of A_{10}, A_{12} and A_{22} we see that

$$2(A_{12}+A_{12}a_0)\equiv0,\quad 2(A_{22}+A_{22}a_0)\equiv0,\quad 2(A_{10}+A_{10}a_0)\equiv0. \tag{6}$$

From $2\beta\equiv2(A_{01}+A_{21})a_1+2(A_{03}+A_{23})a_1a_2$, we have

$$2(A_{01}+A_{21})\equiv0,\quad 2(A_{03}+A_{23})\equiv0. \tag{7}$$

Next,

$$2\gamma\equiv2(A_{01}+A_{02}+A_{03})a_0+2[A_{02}+A_{22}+A_{12}+A_{32}+a_0(A_{12}+A_{21}+A_{03}+A_{22}+A_{32})]a_1.$$

Hence, in view of (7), and (6),

$$2(A_{01}+A_{02}+A_{03})a_0\equiv0,\quad 2(A_{02}+A_{32})(1+a_0)\equiv0. \tag{8}$$

Making use of the above relations, $\beta+\gamma+\delta\equiv 0$ gives

$$\begin{aligned}
&2A_{10}a_0+2A_{12}a_0+A_{01}a_0+A_{02}a_0^2+A_{03}a_0^3+a_2(2A_{02}a_0+2A_{12}a_0+3A_{03}a_0^2)+3A_{03}a_0a_2^2\\
&\quad+[A_{01}+2A_{02}a_0+2A_{22}a_0+2A_{30}a_0+2A_{32}a_0+A_{12}a_0^2+3A_{03}a_0^2+a_2(2A_{02}+2A_{32})\\
&\quad+3A_{03}a_2^2]a_1+[A_{02}+A_{22}+A_{21}a_0+3A_{03}a_0+A_{22}a_0^2+3A_{23}a_0+A_{23}a_0^3\\
&\qquad\qquad\qquad+a_2(3A_{03}+3A_{23}+3A_{23}a_0^2)+3A_{23}a_0a_2^2]a_1^2\\
&\quad+[A_{21}+A_{12}+A_{32}+A_{03}+A_{23}+3A_{23}a_0^2+A_{32}a_0^2+3A_{23}a_2^2]a_1^3\equiv 0.
\end{aligned}$$

Denote this congruence by $p+qa_1+ra_1^2+sa_1^3\equiv 0$. Hence $p\equiv 2q\equiv 2r\equiv q+r+s\equiv 0$. From $p\equiv 0$ we have

$$\left.\begin{aligned}
&2A_{10}a_0+2A_{12}a_0+A_{01}a_0+A_{02}a_0^2+A_{03}a_0^3\equiv 0,\quad 2A_{03}a_0\equiv 0,\\
&2A_{02}a_0+2A_{12}a_0+A_{03}a_0^2+A_{03}a_0\equiv 0.
\end{aligned}\right\} \tag{9}$$

From $2q\equiv 0$ we have

$$2(A_{01}+A_{12}a_0)\equiv 0,\quad 2A_{03}\equiv 0. \tag{10}$$

By (7) we have $2A_{23}\equiv 0$. Hence $A_{23}a_1^2a_2^3\equiv A_{23}a_1^2a_2^2$. Hence we may take $\alpha_{23}\equiv 0$ by replacing ϕ by $\phi+\alpha_{23}S$, and $\gamma_{23}\equiv 0$ by subtracting $\gamma_{23}Q$. $A_{23}a_1^2a_2^2$ is now in a form that can be combined with $A_{22}a_1^2a_2^2$. Hence we may assume $A_{23}\equiv 0$. Similarly, by combining $A_{03}a_2^3$ with $A_{02}a_2^2$ we may take $A_{03}\equiv 0$. It is readily verified that the above reductions do not disturb the quantities that we have previously taken to be zero. A like remark applies to the reductions that follow.

Since $A_{01}\equiv\alpha_{01}$, we have, from (10), that $2A_{01}\equiv 0$. Take $A_{01}=0$ by combining $A_{01}a_2\equiv A_{01}a_2^2$ with $A_{02}a_2^2$. From (7) we now have that $2A_{21}\equiv 0$. Hence $A_{21}a_1^2a_2\equiv A_{21}a_1^2a_2^2$ and we may take $\alpha_{21}\equiv 0$ by considering $\phi+\alpha_{21}S$. We now have $A_{21}a_1^2a_2\equiv\delta_{21}a_0^3a_1^2a_2^2$ and may combine it with $A_{22}a_1^2a_2^2$ and take $A_{21}\equiv 0$. From (10) and (6) we have $2A_{12}\equiv 0$. Hence, combining $A_{12}a_1a_2^2\equiv A_{12}a_1^2a_2^2$ with $A_{22}a_1^2a_2^2$, we may take $A_{12}\equiv 0$. The above conditions now become

$$\left.\begin{aligned}
&2A_{02}+2A_{32}(1+a_0)\equiv 0,\quad 2A_{10}a_0+A_{02}a_0^2\equiv 0,\\
&A_{01}\equiv A_{03}\equiv A_{12}\equiv A_{21}\equiv A_{23}\equiv 0,\quad 2(A_{22}+A_{22}a_0)\equiv 0.
\end{aligned}\right\} \tag{11}$$

From $2r\equiv 0$ we have $2(A_{02}+A_{22}+A_{22}a_0)\equiv 0$, whence $2A_{02}\equiv 0$. From $q+r+s\equiv 0$ we have

$$2A_{22}a_0+2A_{30}a_0+2A_{32}a_0+A_{02}+A_{22}+A_{22}a_0^2+A_{32}+A_{32}a_0^2+2A_{32}a_2\equiv 0. \tag{12}$$

Hence $2A_{32}\equiv 0$. We may take $\alpha_{32}\equiv 0$ by considering $\phi+\alpha_{32}S$. By combining $A_{32}a_1^3a_2^2\equiv\delta_{32}a_0^3a_1^2a_2^2$ with $A_{22}a_1^2a_2^2$ we have $A_{32}\equiv 0$. By the final condition (11), (12) now becomes

$$A_{02}+A_{22}-A_{22}a_0^2+2A_{30}a_0\equiv 0. \tag{13}$$

From (11) we have

$$2\alpha_{02}\equiv 2(\beta_{02}+\delta_{02})\equiv 2\gamma_{02}\equiv 2\delta_{10}+\alpha_{02}+\beta_{02}+\gamma_{02}+\delta_{02}\equiv 0. \tag{14}$$

From (13) we obtain the additional conditions

$$\alpha_{02} \equiv 2(\beta_{02}+\beta_{22}) \equiv 2(\alpha_{30}+\delta_{30})+\beta_{02}+\gamma_{02}+\delta_{02} \equiv 0. \tag{15}$$

From (14) and (15) we have

$$\alpha_{02} \equiv 2\gamma_{02} \equiv 2\delta_{10}+\beta_{02}+\gamma_{02}+\delta_{02} \equiv 2(\beta_{02}+\beta_{22}) \equiv 2(\delta_{10}+\alpha_{30}+\delta_{30}) \equiv 0. \tag{16}$$

We now have

$$\begin{aligned}\phi = \alpha_{00}&+\beta_{00}a_0+\gamma_{00}a_0^2+\delta_{00}a_0^3+\beta_{02}a_0a_2^2+\gamma_{02}a_0^2a_2^2+\delta_{02}a_0^3a_2^2+\delta_{10}a_0^3a_1\\ &+\beta_{20}a_0a_1^2+\gamma_{20}a_0^2a_1^2+\delta_{20}a_0^3a_1^2+\beta_{22}a_0a_1^2a_2^2+\delta_{22}a_0^3a_1^2a_2^2+\alpha_{30}a_1^3+\delta_{30}a_0^3a_1^3,\end{aligned}$$

where the Greek letter constants satisfy relations (16). Let ϕ become ϕ' under the replacement (4). Then $\phi-\phi'=p+qa_2+ra_2^2+sa_2^3$, where

$$\begin{aligned}p &= \beta_{00}a_0+\gamma_{00}a_0^2+\delta_{00}a_0^3+\delta_{10}a_0^3a_1+2\alpha_{30}a_1^3+\delta_{30}a_0^3a_1^3+\beta_{20}a_0a_1^2+\gamma_{20}a_0^2a_1^2+\delta_{20}a_0^3a_1^2,\\ q &= -\beta_{00}-\beta_{02}a_0^2-\beta_{20}a_1^2-\beta_{22}a_0^2a_1^2,\\ r &= \beta_{02}a_0+\delta_{02}a_0^3-\gamma_{00}+\beta_{22}a_0a_1^2+\delta_{22}a_0^3a_1^2-\gamma_{20}a_1^2,\\ s &= -\delta_{00}-\delta_{02}a_0^2+\delta_{10}a_1-\delta_{20}a_1^2-\delta_{22}a_0^2a_1^2+\delta_{30}a_1^3.\end{aligned}$$

Then $p \equiv 0$ gives $\beta_{00} \equiv \gamma_{00} \equiv \delta_{00} \equiv 0$ and

$$2\delta_{10} \equiv 2(\beta_{20}+\gamma_{20}+\delta_{20}) \equiv 2\delta_{30} \equiv 0, \quad 2\alpha_{30}+\beta_{20}a_0+\gamma_{20}a_0^2+(\delta_{10}+\delta_{20}+\delta_{30})a_0^3 \equiv 0,$$

whence

$$2\alpha_{30} \equiv 2\beta_{20} \equiv 2\delta_{20} \equiv 2\gamma_{20} = 0.$$

Hence $A_{20} \equiv 2N_{20}a_0^3$. Take $A_{20} \equiv 0$ by combining $2N_{20}a_0^3a_1^2 \equiv 2N_{20}a_0^3a_1$ with $\delta_{10}a_0^3a_1$. Also take $\alpha_{30} \equiv 0$ by considering $\phi-\alpha_{30}I$. The remaining condition from $p \equiv 0$ is $\delta_{10} \equiv \delta_{30}$. Hence ϕ has the term $\delta_{10}(a_0^3a_1+a_0^3a_1^3) \equiv 0$. Thus ϕ is congruent to an expression with $\delta_{10} \equiv \delta_{30} \equiv 0$. From $2r \equiv 0$ we have $2(\beta_{22}+\delta_{22}) \equiv 0$. From $q+r+s \equiv 0$ we have

$$\beta_{02}a_0-(\beta_{02}+\delta_{02})a_0^2+\delta_{02}a_0^3+[\beta_{22}a_0-(\beta_{22}+\delta_{22})a_0^2+\delta_{22}a_0^3]a_1^2 \equiv 0,$$

whence

$$2\beta_{02} \equiv 2\delta_{02} \equiv 2\beta_{22} \equiv 2\delta_{22} \equiv 0.$$

From the first two of these, together with $2\gamma_{02} \equiv 0$ and $\beta_{02}+\gamma_{02}+\delta_{02} \equiv 0$, from (16), (since $2\delta_{10} \equiv 0$), we have $A_{02} \equiv 0$. From $2\beta_{22} \equiv 2\delta_{22} \equiv 0$ we have

$$\beta_{22}a_0a_1^2a_2^2+\delta_{22}a_0^3a_1^2a_2^2 \equiv (\beta_{22}+\delta_{22})a_0^2a_1^2a_2^2 \equiv (\beta_{22}+\delta_{22})Q,$$

and hence $\phi \equiv \alpha_{00}+(\beta_{22}+\delta_{22})Q$. From this we have

Theorem 5. *Any modular invariant of a binary quadratic form whose coefficients are integers modulo 4 is a linear function of the invariants I, H, Q, J, K, E.*

The binary form is congruent to a multiple of a square of a linear form if and only if $I \equiv K \equiv E \equiv 0 \pmod 4$.

*The Ten Nodes of the Rational Sextic and of the Cayley Symmetroid.**

By Arthur B. Coble.†

Introduction.

The general rational plane sextic with ten nodes occupies a unique position among all rational plane curves in that it is the rational curve of lowest order which can not be transformed by ternary Cremona transformation into a straight line, that is to say its order can not be reduced by such transformation. It may, however, be transformed into other rational sextics, and this can be accomplished by Cremona transformations of infinitely many distinct types. One of the principal results of this paper is that the sextic and all of its sextic transforms are comprised under precisely $2^{13}.31.51$ projectively distinct types.

The intimate relation between the ten nodes of a rational plane sextic and the ten nodes of that quartic surface known as the Cayley symmetroid has been pointed out by J. R. Conner.‡ It is not surprising therefore to find that a similar fact is true of the symmetroid under regular Cremona transformation in space.

The methods of investigation here employed have been set forth in an earlier series of papers by the writer.§ Some of the points of view may be recapitulated briefly as follows. We shall be interested in a Cremona transformation C only in so far as it disturbs projective relations so that for our purposes $C \equiv \pi C \pi'$ where π, π' are arbitrary projectivities. If C has the singular points, or F-points, $p_1, \ldots, p_\rho$, and C^{-1} the F-points $q_1, \ldots, q_\rho$ then C transforms curves of order x_0 and multiplicities x_i at p_i into curves of order x_0' and multiplicities x_j' at q_j $(i, j=1, \ldots, \rho)$ where x' is determined in terms of x by the linear transformation, $L(C)$,

$$(1) \qquad L(C): \; x_0'=mx_0-\Sigma_{i=1}^{i=\rho} r_i x_i, \quad x_j'=s_j x_0-\Sigma_{i=1}^{i=\rho} \alpha_{ij} x_i.$$

* Read by title at the meeting of the Chicago Section of the American Mathematical Society, April, 1919.

† This investigation has been carried on under the auspices of the Carnegie Institution of Washington, D. C.

‡ "The Rational Sextic Curve and the Cayley Symmetroid," this Journal, Vol. XXXVII (1915), p. 29.

§ "Point Sets and Cremona Groups," Part II, *Trans. Amer. Math. Soc.*, Vol. XVII (1916), p. 345; referred to hereafter as P. S. II.

In (1) the coefficients are m, the order of C; r_i, the order of F-point p_i; s_j, the order of the F-point q_j; and α_{ij}, the number of times the fundamental curve, or F-curve, of p_i passes through q_j.

The product CC' of two Cremona transformations can be unique only when the position of the F-points p'_i of C' with respect to the F-points q_j of C^{-1} is definitely specified. In order to limit the possibilities which arise in this connection we require that the points p_i shall be in a set of n points P_n^2 and the points q_j in a set of n points Q_n^2 such that the further pairs $p_{\rho+1}$, $q_{\rho+1}, \ldots, p_n, q_n$ are pairs of ordinary corresponding points of C. This amplifies the linear transformation $L(C)$ by the equations

$$(2) \qquad x'_l = -(-1)x_l \quad (l=\rho+1, \ldots, n),$$

and the two sets P_n^2, Q_n^2 are called *congruent under C*. In forming the product CC' we require that the points of $P_n'^2$ shall coincide *in some order* with the points of Q_n^2. This possibility of reordering the points of a set—a non-projective operation for $n>4$—is accounted for by adjoining to the linear transformations $L(C)$ those additional ones constituting a $g_{n!}$ which permute the variables $x_1, \ldots, x_n$. Thus the operations involved in passing from a set P_n^2 to all sets Q_n^2 congruent in some order to P_n^2—operations which constitute a group $G_{n,2}$—are reflected by simple isomorphism in the transformations $L(C)$ of the group $g_{n,2}$ generated by $g_{n!}$ and the transformation $L(C)$ determined by a single quadratic transformation C, since the general Cremona transformation is a product of properly ordered quadratic transformations. Obviously when a set P_n^2 is in question this general transformation is restricted to have $\rho \leqq n$ F-points.*

We are concerned here with the set P_{10}^2 of the nodes of a rational plane sextic and can state at once the theorem

(3) *A sextic S with nodes P_{10}^2 can be transformed into a sextic $\bar{S}$ with nodes Q_{10}^2 by ternary Cremona transformation if and only if the sets P_{10}^2 and Q_{10}^2 are congruent.*

For if S is transformed by C into $\bar{S}$ the ρ F-points of C must be all within P_{10}^2 else the order of the transform is greater than 6. Hence $\rho \leqq 10$. If $\rho < 10$ the nodes of S in P_{10}^2 which are ordinary points of C pass into nodes of $\bar{S}$ in the congruent set Q_{10}^2.

The arithmetic group $g_{n,2}$ simply isomorphic with $G_{n,2}$ has integer coefficients. We shall prove in § 1 that there is only a finite number of projectively distinct sets Q_{10}^2 congruent to the set P_{10}^2 *when P_{10}^2 is the set of nodes of S*, and that, for all the operations of $G_{10,2}$ whose isomorphic elements in

* These remarks are amplified in P. S. II, § 1.

$g_{10,2}$ have coefficients congruent modulo 2 to those of the identity, the set Q_{10}^2 is projective to P_{10}^2 and therefore may be made to coincide with P_{10}^2 by a subsequent projectivity. These elements form an invariant subgroup $\bar{g}_{10,2}$ of $g_{10,2}$ whose factor group $g_{10,2}^{(2)}$* is finite and of order $10!\,2^{18}.31.51$.

An important problem is now apparent. Since $g_{10,2}$ is infinite and discontinuous (P. S. II, § 4 (18)) and $\bar{g}_{10,2}$ is of finite index under $g_{10,2}$ there follows that an infinite discontinuous Cremona group $\bar{G}_{10,2}$ exists which transforms the sextic S into itself. $\bar{G}_{10,2}$ also will contain an invariant subgroup $\bar{\bar{G}}_{10,2}$ which consists of those elements of $\bar{G}_{10,2}$ for which every point of S is fixed. It may be and probably is true that $\bar{\bar{G}}_{10,2}$ is merely the identical transformation, but in any case the factor group of $\bar{\bar{G}}_{10,2}$ under $\bar{G}_{10,2}$ will be represented by a discontinuous group of elements of the form

$$t' = \frac{at+b}{ct+d},$$

where t is the parameter on the rational curve S. From certain geometrical considerations it seems reasonable to think that this discontinuous group is of genus 4, and that the ten nodes of S can be expressed by means of Riemannian modular functions of genus 4.

The ten nodes P_{10}^3 of the Cayley symmetroid Σ, discussed in Part II, behave under *regular* † Cremona transformations in space much like the ten nodes of S under ternary transformation. One novelty introduced in § 4 is the *dilation* of the regular group in a space S_k into a subgroup of the regular group in a higher space S_{k+l}.

PART I.

The Ten Nodes P_{10}^2 of the Sextic S.

§ 1. *The Equivalence of the f-curves of P_{10}^2 under $\bar{G}_{10,2}$.*

The first theorem which we shall use is

(4) *The group $\bar{G}_{10,2}$ which leaves the sextic S unaltered is generated by the involutions conjugate under $G_{10,2}$ to the Bertini involution.*

We recall that the Bertini involution is defined as follows. Given eight points $p_1, \ldots, p_8$ in the plane, the ∞^3 sextics with nodes at these points have the property that the ∞^2 sextics of the system on a point x pass also through another point y, the copoint of x in the involution B. Obviously every sextic

* The factor groups $g_{n,k}^{(2)}$ for the group $g_{n,k}$ have been identified with known groups in the author's paper entitled "Theta Modular Groups Determined by Point Sets," this Journal, Vol. XL (1918), p. 317; cited hereafter as T. M. Groups. This paper emphasizes the geometric possibilities of the particular cases $g_{2p+2,p}$. It is of interest to find that other cases also have geometric applications.

† Cf. P. S. II, § 4, or § 4 of this paper for the definition.

of the system is a fixed curve, and every additional node of such a sextic is a fixed point of the involution whence it leaves the sextic S with nodes at $p_1, \ldots, p_8, p_9, p_{10}$ unaltered. By permutation of the points of P_{10}^2 all the $\binom{10}{8}$ Bertini involutions attached to the set P_{10}^2 are obtained. Moreover, if C is any Cremona transformation with F-points at P_{10}^2, then CBC^{-1} also leaves S unaltered. For C transforms S into a sextic S' with nodes at Q_{10}^2, B leaves S' unaltered, and C^{-1} transforms S' back into S. Hence the conjugate set of involutions described in (4) all belong to $\bar{G}_{10,2}$. The proof that they generate $\bar{G}_{10,2}$ will appear later. Meanwhile two objects conjugate under $\bar{G}_{10,2}$ will be called equivalent, and this relation of equivalence will be denoted by the symbol $\equiv$.

The f-curves of the set P_{10}^2 are the transforms by Cremona transformation of the sets of directions about the points. Instead of the general Cremona transformation we may make repeated use of the quadratic transformation $A_{j_1j_2j_3}$ with F-points at $p_{j_1}, p_{j_2}, p_{j_3}$. Beginning then with the set of directions about p_1, it becomes under the $g_{n!}$ of permutations of the points, a set of directions about any one of the ten points. Applying A_{123} to the set of directions at p_1 it becomes the line on q_2q_3, and under $g_{n!}$ this becomes any line q_iq_j. Applying A_{123} to the line p_4p_5 it becomes a conic on $q_1q_2q_3q_4q_5$. Proceeding in this way the totality of f-curves of the set P_{10}^2 is obtained. We shall denote by its *signature*, $f_r(j_1^{k_1}, j_2^{k_2}, \ldots, j_{10}^{k_{10}})$, an f-curve of order r with multiple points of orders $k_1, \ldots, k_{10}$ at the points $p_1, \ldots, p_{10}$, respectively. A systematic derivation of the types of f-curves is carried out in the following table (5):

	f-curve	operated upon by	becomes	which is	
	$f_0(1)$	A_{123}	$f_1(23)$*		
		A_{234}	$f_0(1)$		
	$f_1(23)$	A_{123}	$f_0(1)$		
		A_{124}	$f_1(23)$		
		A_{145}	$f_2(12345)$*		
	$f_2(12345)$	A_{345}	$f_1(12)$		
		A_{456}	$f_2(12345)$		
		A_{567}	$f_3(12345^267)$*		
(5)		A_{678}	$f_4(123456^27^28^2)$	$\equiv f_2(12345)$	(1°)
	$f_3(12345^267)$	A_{567}	$f_2(12345)$		
		A_{467}	$f_3(12345^267)$		
		A_{578}	$f_3(12345^267)$		
		A_{678}	$f_4(12345^26^27^28)$	$\equiv f_2(12348)$	(1°)
		A_{589}	$f_4(12345^36789)$*		
		A_{789}	$f_5(12345^267^38^29^2)$	$\equiv f_3(12345^267)$	(2°)
		A_{8910}	$f_6(12345^2678^39^310^3)$	$\equiv f_4(1234678^3910)$	(3°)
	$f_4(12345^36789)$	A_{567}	$f_3(12345^289)$		
		A_{678}	$f_5(12345^36^27^28^29)$	$\equiv f_3(123456^29)$	(2°)
		A_{5610}	$f_4(12345^36789)$		
		A_{6710}	$f_6(12345^36^37^38910^2)$	$\equiv f_4(12345^36789)$	(3°)

New types of f-curves as they are obtained are starred, and these new types are in turn subjected to transformation. However, as the process goes on, the new types obtained are equivalent under $\overline{G}_{10,2}$ to earlier types and these need not be transformed afresh.

In order to prove the equivalences (1^0), (2^0), (3^0) listed in the table (5), and at the same time to verify that the two further equivalences

$$(4^0) \qquad f_3(i_1i_2i_3i_4i_5i_6j^2) \equiv f_3(i_1i_2i_3i_4i_5i_6k^2),$$

$$(5^0) \qquad f_4(i_1i_2i_3i_4i_5i_6i_7jk^3) \equiv f_4(i_1i_2i_3i_4i_5i_6i_7j^3k),$$

are valid we begin with the equivalence,

$$(6) \qquad f_0(i) \equiv f_6(i^3j_1^2j_2^2j_3^2j_4^2j_5^2j_6^2j_7^2)$$

which is derived at once from a Bertini involution. If the two members of this equivalence be transformed by C then the two transforms are themselves equivalent under the transform of the Bertini involution by C, whence according to (4) they are equivalent under $\overline{G}_{10,2}$. Transforming (6) by $A_{ij_1j_2}$, $A_{ij_3j_4}$, $A_{ij_5j_6}$ and $A_{ij_7j_8}$ successively we get

$$(7) \qquad f_1(j_1j_2) \equiv f_5(i^2j_1j_2j_3^2j_4^2j_5^2j_6^2j_7^2),$$

$$(1^0) \qquad f_2(ij_1j_2j_3j_4) \equiv f_4(ij_1j_2j_3j_4j_5^2j_6^2j_7^2),$$

$$(4^0) \qquad f_3(i^2j_1j_2j_3j_4j_5j_6) \equiv f_3(j_1j_2j_3j_4j_5j_6j_7^2),$$

$$(5^0) \qquad f_4(i^3j_1j_2j_3j_4j_5j_6j_7j_8) \equiv f_4(ij_1j_2j_3j_4j_5j_6j_7^3j_8).$$

If now we transform (4^0) by $A_{j_1j_7j_8}$ and $A_{j_7j_8j_9}$ we get

$$(2^0) \qquad f_5(i^2j_1^3j_2j_3j_4j_5j_6j_7^2j_8^2) \equiv f_3(j_1j_2j_3j_4j_5j_6j_7^2),$$

$$(3^0) \qquad f_6(i^2j_1j_2j_3j_4j_5j_6j_7^3j_8^3j_9^3) \equiv f_4(j_1j_2j_3j_4j_5j_6j_7^3j_8j_9),$$

whence all the equivalences used in limiting the table (5) have been established.

A glance at the list of equivalences established shows that the signatures of equivalent f-curves are congruent modulo 2, and further that no two of the non-equivalent f-curves in the first column of table (5) have signatures which are congruent modulo 2. This is to be expected since the signatures, $f_{r_i}(p_1^{a_{i1}}, p_2^{a_{i2}}, \ldots, p_{10}^{a_{i10}})$, of the f-curves of P_{10}^2 arise from the columns other than the first of the matrices of the linear transformations L of (1) and (2), and the transformation L which correspond to the Bertini involutions (and therefore also to the conjugates of the Bertini involutions) are congruent to the identity modulo 2. Thus we have proved that

(8) *Under the group generated by the conjugate set of involutions which contains a Bertini involution, the infinite number of f-curves of P_{10}^2 divide into $527 = 2^{p-1}(2^p+1)-1\,(p=5)$ sets such that the infinite number in*

any one set are equivalent and that the f-curves from different sets are not equivalent. Equivalent f-curves have signatures congruent modulo 2. As types of these sets we may take the $\binom{10}{1}$ *of form* $f_0(i)$, *the* $\binom{10}{2}$ *of form* $f_1(i_1 i_2)$, *the* $\binom{10}{5}$ *of form* $f_2(i_1 i_2 i_3 i_4 i_5)$, *the* $\binom{10}{8}$ *of form* $f_3(i_1^2 i_2 i_3 i_4 i_5 i_6 i_7)$, *and the* $\binom{10}{9}$ *of form* $f_4(i_1^3 i_2 i_3 i_4 i_5 i_6 i_7 i_8 i_9)$.

Since all f-curves with signatures congruent modulo 2 are equivalent under the conjugates of a Bertini involution, there follows that the subgroup $g(2)$ of $g_{10,2}$ which is congruent modulo 2 to the identity is generated by these conjugates. Now the index of $g(2)$ under $g_{10,2}$ is the order of the finite group of permutations, $g_{10,2}^{(2)}$, of the above 527 sets. The order of this group has been determined in "T. M. Groups." In fact the signatures of the f-curves reduce modulo 2 to the coefficients of the forms b_1, c_2 of the table there given (p. 323 for $\nu=\kappa=2$). They are permuted like the even characteristics of the theta functions for $p=5$ under the group (p. 337 *loc. cit.*) of order

$$\mu=2^{21}(2^5-1)(2^8-1)(2^6-1)(2^4-1)(2^2-1),$$

which leaves one even theta characteristic unaltered. Now $g(2)$ is simply isomorphic either with $\overline{G}_{10,2}$ or with a subgroup of it. In the first case μ divided by 10! (to account for the mere ordering of the set P_{10}^2) will be the number of sextics projectively distinct from S and including S itself. In the second case this number will be a smaller factor of $\mu/10!$ Now assuming that μ is the proper index of $\overline{G}_{10,2}$ under $G_{10,2}$ then the index μ' of $\overline{G}_{9,2}$ under $G_{9,2}$ (where these new groups are defined precisely as the groups $\overline{G}_{10,2}$ and $G_{10,2}$ except that all Cremona transformations employed are to have p_{10} as an ordinary point*) is $\mu/527$, since p_{10} or $f_0(10)$ is to be unaltered. Then the number of projectively distinct sets P_9^2 which can be obtained by Cremona transformation from the nine nodes of a sextic is $\mu/527$ divided by 9! But according to P. S. II (47) $\mu/9!527=2^9.960$ is precisely this number of sets P_9^2. Hence μ is the index of $\overline{G}_{10,2}$ under $G_{10,2}$ and $\overline{G}_{10,2}$ is generated by the conjugates of the Bertini involution. We have thus completed the proof of (4) and have also proved that

(9) *A rational plane sextic with ten nodes can be transformed by Cremona transformation into precisely* $2^{18}.31.51$ *projectively distinct sextics. Under such transformation these projectively distinct types (with*

*It is proved in P. S. II, § 6, that the generators of $\overline{G}_{9,2}$ are conjugates of Bertini involutions whence all the Cremona transformations with F-points at P^2_9 for which P^2_9 is congruent to itself will leave unaltered the 10-th node of a sextic with nodes at P^2_9.

ordered nodes) are permuted according to the finite group of odd and even theta characteristics for $p=5$, which leaves an even characteristic unaltered. The infinite discontinuous group $\bar{G}_{10,2}$ of Cremona transformations which leaves S unaltered is simply isomorphic with the subgroup $\bar{g}_{10,2}$ of $g_{10,2}$, which is congruent to the identity modulo 2.

§ 2. *The Discriminant Conditions for P_{10}^2.*

In P. S. II, § 8, the set P_7^2 was discussed in connection with the general plane quartic and the sixty-three factors of the discriminant of this quartic arose from the conditions that two points of P_7^2 should coincide, that three should be on a line, and that six should be on a conic. In all these cases an f-curve passes through one more point of the set than is true in general. The conditions might be indicated thus:

$$f_0(i_1 i_2)=0 \quad f_1(i_1 i_2 i_3)=0, \quad \text{and} \quad f_2(i_1 i_2 i_3 i_4 i_5 i_6)=0.$$

Similarly for the set P_6^2 (P. S. III (1917), § 1), the same conditions give rise to the thirty-six factors of the discriminant of the cubic surface which is mapped from the plane by cubic curves on P_6^2. We shall therefore continue to refer to such conditions as *discriminant conditions for the set*, even though for sets beyond P_8^2 the word discriminant does not have its usual meaning.

For a general set P_{10}^2 the number of these discriminant conditions is infinite, but they all arise from any one—say $f_0(12)=0$—by Cremona transformation. On the other hand when P_{10}^2 is the special set of ten nodes of a sextic S and therefore subject to three conditions, the existence of one discriminant condition—a fourth condition on P_{10}^2—taken together with the three conditions already implied by the existence of S entails the existence of infinitely many discriminant conditions. For example, reverting to the table (5) of § 1, let us begin with the condition $f_0(1,9)=0$ which indicates the existence of a tacnode due to the coincidence in some direction of the nodes p_1, p_9. Transforming this by A_{123} we get the condition $f_1(239)=0$ which expresses that the nodes p_2, p_8, p_9 are on a line. Transforming this by A_{145} we get the condition $f_2(123459)=0$, and this, transformed by A_{678} gives rise to $f_4(123456^2 7^2 8^2 9)=0$. But according to the equivalence (1^0) there is a transformation of $\bar{G}_{10,2}$ which leaves the nodes of S unaltered and transforms $f_4(123456^2 7^2 8^2)$ into $f_2(12345)$. Therefore if $f_4(123456^2 7^2 8^2 9)=0$ then also $f_2(123459)=0$. Proceeding thus we find that the equivalences of f-curves under $\bar{G}_{10,2}$ imply the identity of corresponding discriminant conditions and we can prove at once by the foregoing methods the theorem:

(10) *The number of discriminant conditions—infinite for the general point set P_{10}^2—is finite for the P_{10}^2 of nodes of S, a set subject to three conditions and containing nine absolute constants. Any two discriminant conditions whose signatures are congruent modulo 2 impose the same* FOURTH *condition on the ten nodes. The $\binom{10}{2}$ conditions of type $f_0(i_1 i_2)=0$, the $\binom{10}{3}$ of type $f_1(i_1 i_2 i_3)=0$, the $\binom{10}{6}$ of type $f_2(i_1 i_2 i_3 i_4 i_5 i_6)=0$, the $\binom{10}{7}$ of type $f_3(i_1^2 i_2 i_3 i_4 i_5 i_6 i_7 i_8)=0$, and the $\binom{10}{10}$ of type $f_4(i_1^3 i_2 i_3 i_4 i_5 i_6 i_7 i_8 i_9 i_{10})=0$, $496=2^{p-1}(2^p-1)$ $(p=5)$ in all, exhaust the number of independent discriminant conditions. The members of this finite set of conditions are permuted under Cremona transformation like the odd theta characteristics under the group of* §1 (9).

In fact these conditions correspond to the forms b_2, c_8 of the table cited above from T. M. Groups.

From any equivalence there will follow a theorem concerning a special sextic S. Thus from (4°) and (5°) of §1 we have

(11) *If there exists a cubic curve on seven nodes of S with a double point at one of the three remaining nodes (one condition on S) then there will exist a cubic curve on the same seven nodes and with a double point at* ANY *one of the three remaining nodes.*

(12) *If there exists a quartic curve with triple point at one node of S and on the other nodes, then there will exist a quartic with a triple point at any one node and on the other nodes.*

Part of the content of theorem (10) has been stated by Miss Hilda Hudson,* and her method (Section 4, *loc. cit.*) of proving the equivalence of discriminant conditions is interesting. Unfortunately much of this paper is colored by the false assumption that the rational sextic with which she begins, and which has six nodes on a conic is a general rational sextic with nine absolute constants. Miss Hudson uses a space sextic of genus 4—the complete intersection of a quadric and a cubic surface—and assigns to it four actual nodes by making the cubic touch the quadric at four points, and projects it from an arbitrary point of space. Now if λ, μ are the binary parameters of the generators on the quadric, the sextic of genus 4 has the equation $(a\lambda)^3(b\mu)^3=0$ with fifteen constants. Of these six can be removed by projectivities on λ, μ whence the curve has nine absolute constants. These are in

* "The Cremona Transformations of a Certain Plane Sextic," *Proceedings of the London Mathematical Society*, Ser. 2, Vol. XV (1916–17), p. 385.

fact its Riemannian moduli since the curve is normal. The four node requirement reduces the number of constants to 5, and projection from an arbitrary point introduces three more, so that the resulting rational sextic has but eight absolute constants and is subject to the further condition that six nodes are on a conic—a well-known condition on the nodes of any projection of the general space sextic of genus 4. The general rational plane sextic should be obtained as the projection of a general rational space sextic, and the latter sextic does not lie on a quadric.

In the same volume of the *Proceedings* Mr. J. Hodgkinson* shows that there can be *at most* thirty rational sextics with nine properly assigned nodes. As a matter of fact this number is exactly twelve.

In view of these misconceptions it may be worth while to develop in some detail the conditions on the nodes of a rational sextic.† Let then $p_1, \ldots, p_8$ be eight general points of the plane with eight absolute constants. They are the base points of a pencil of cubics $C_\lambda = \lambda_1 C_1 + \lambda_2 C_2$ which meet again in a 9-th point P. This is of course a general pencil of cubics, and all of its members are nondegenerate and all are elliptic except for the twelve nodal cubics of the pencil with nodes at $D_1, \ldots, D_{12}$. The net of sextics, $\mu_1 C_1^2 + \mu_2 C_1 C_2 + \mu_3 C_2^2$, has nodes at $p_1, \ldots, p_8$ and is merely the aggregate of pairs of the pencil C_λ. Other sextics with these nodes exist. Such for example is the degenerate sextic $f_1(12) \cdot f_5(123^2 \ldots 8^2)$ whose factors are known to exist and to be unique. Moreover, this sextic is not found in the above net since it is not a pair of cubics of the pencil C_λ. Let then Σ be any sextic, not included in the net, which has double points at $p_1, \ldots, p_8$. The web of sextics

$$(13) \qquad \mu_1 C_1^2 + \mu_2 C_1 C_2 + \mu_3 C_2^2 + \mu_4 \Sigma$$

contains *all* sextics with nodes at $p_1, \ldots, p_8$. For if another sextic Σ' not contained in the system (13) should exist, the system of ∞^4 sextics obtained by adjoining Σ' would cut the line $f_1(12)$ in ∞^3 variable pairs and a pencil would have the fixed factor $f_1(12)$ and the variable factor $f_5(123^2 \ldots 8^2)$ contrary to the fact that this quintic is unique.

All the sextics of the web (13) on a point x pass through a second point y, and x, y are partners in the Bertini involution B.‡ In fact, if C_1 is the cubic

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* "The Nodal Points of a Plane Sextic," *loc. cit.*, p. 343.

† Cf. E. C. Valentiner, *Tidsskrift for Math.*, Ser. 4, Vol. V (1881), p. 88, and G. Halphen, *M. S. F. Bull.*, Vol. X (1882), p. 162.

‡ Cf. V. Snyder, "The Involutorial Birational Transformation of the Plane of Order Seventeen," this Journal, Vol. XXXIII (1911), p. 327.

of the pencil C_λ on x, then C_1^2 and C_1C_2 are two independent sextics on x; let Σ be a third. These sextics all meet at the intersections of C_1 and Σ. Let the elliptic parameters on C_1 of $p_1, \ldots, p_8$ be $u_1, \ldots, u_8$ (with $u+u'+u''\equiv 0$ as the linear condition) and let u_x, u_y be those of x, y. Then

$$2(u_1+\ldots+u_8)+u_x+u_y\equiv 0.$$

Hence x, y are on a line with the point $u=2(u_1+\ldots+u_8)$ and this is the tangential point of the four points $u=-(u_1+\ldots+u_8)+\frac{\omega}{2}$. If $\frac{\omega}{2}$ is the zero half-period, this is the 9-th base point P; if $\frac{\omega}{2}$ is a proper half-period we we may call the points the *three half-period points on* C_1. Hence a construction for B is as follows: At P, a base point of the pencil C_λ, draw a tangent to the cubic C_λ to meet the cubic C_λ at P_λ, and from P_λ project the cubic into itself to obtain the pairs x, y of B. One easily verifies that the locus of P_λ is a rational quartic on $p_1, \ldots, p_8$ with triple point at P whose tangents are those of cubics with flexes at P. The construction for y is indeterminate only when x is at p_1, or p_2, or $\ldots$, or p_8. The sextic S_{p_1} with triple point at p_1 and nodes at $p_2, \ldots, p_8$ exists and is unique (as is proved at once by reducing its order by a quadratic transformation), and, if x is at any point of S_{p_1}, y is at p_1. Hence B has eight six-fold F-points p_i with corresponding f-curve S_{p_i} and is of order 17. Evidently every sextic (13) and every cubic C_λ is a fixed curve of B.

We are interested primarily in the fixed points of B. These occur at the point P and at the three half-period points on C_λ. The latter run over a locus N which has triple points at p_i with the same tangents as S_{p_i} since these three directions at p_i are self corresponding. Also N is of order 9 since a cubic C_λ meets it in three points outside the eight points p_i. The fixed point P and the fixed point p_9—a general point on N—are of different kinds. P is a fixed point with fixed directions, *i. e.*, a curve K on P is transformed by B into a curve K' on P which touches K. This follows from the fact that P is a fixed point on *every* cubic of the pencil C_λ. On the other hand p_9 is a fixed point on but one cubic C_9 of the pencil C_λ and arises from the coincidence at p_9 in the direction of the tangent T_9 to C_9 at p_9 of a copair x, y of B. Hence this is one fixed direction on p_9, and another is the direction T_{Np_9} of N at p_9, *i. e.*, the direction to a neighboring fixed point. Any curve K on p_9 is transformed by B into a curve K' on p_9 such that the tangents to K and K' at p_9 are harmonic to T_9 and T_{Np_9}.

Every point x of the plane is a double point of at least one sextic of the web, namely of the squared cubic, C_1^2, on it. If x is a double point of a second sextic $\bar{\Sigma}$, and therefore of a pencil, then the net determined by C_1^2, C_1C_2, and $\bar{\Sigma}$ on x have their remaining intersection y at x, which may be at P if $\bar{\Sigma}$ is $C_\lambda^2(\lambda \neq 1)$, but otherwise is a point p_9 on N. Conversely the net of sextics on p_9 being fixed curves have as a common direction that of T_9 which belongs to the coincident pair, and therefore a pencil of the net will have a node at p_9 with nodal tangents harmonic to T_9 and to T_{Np_9}. The pencil contains one cuspidal sextic with tangent T_{Np_9} and one squared cubic C_9^2 with tangent T_9. Hence, disregarding nodes and cusps due to the sextics C_λ^2, and disregarding also the point P, we see that N is the locus of nodes of sextics of the web (13); or also the locus of cusps of sextics of the web; or as an envelope is the locus of cusp tangents; or finally is that 9-ic with triple points at $p_1, \ldots, p_8$ and on $D_1, \ldots, D_{12}$. For a double point of a cubic C_λ is projected into itself from a point of C_λ. An equation of N is the Jacobian, $J(C_1, C_2, \Sigma)=0$.

The curve N is of genus 4 and its canonical series g_6^3 is cut out by the web of adjoints (13). The series cut out by the pencil C_λ, a g_3^1, has for residue with respect to g_6^3 the same g_3^1. Thus N differs from the general curve of genus 4 in that the two series, g_3^1, cut out on the norm curve by the two sets of generators of the quadric on the norm curve have coincided, *i. e.*, its canonical adjoints (13) map N into a space sextic cut out on a quadric cone by a cubic surface. Since the quadric is a cone, N has but eight moduli, the absolute constants of $p_1, \ldots, p_8$. A tangent plane of the quadric cone does not count as a tritangent plane of the sextic since it is rather a reunion of a set of g_3^1 and a set of $g_3^{1'}$. The 120 tritangent planes arise from the 120 degenerate sextics, $\binom{8}{1}$ of type $f_0(1)\cdot S_{p_1}$, $\binom{8}{2}$ of type $f_1(12)\cdot f_5(123^2\ldots 8^2)$, $\binom{8}{5}$ of type $f_2(12345)f_4(123456^27^28^2)$, and $\binom{8}{6}$ of type $f_3(1^2234567)\cdot f_3(2345678^2)$. Since a g_n^r has $(r+1)(n+rp-r)$ $(r+1)$-fold points, g_3^1 has twelve double points which are at $D_1, \ldots, D_{12}$. If p_9 is a general point on N there is as we have seen, a pencil of sextics with a node at p_9. This pencil cuts N in a g_4^1 with fourteen double points. Two of these double points arise from the two further intersections of the squared cubic C_9^2 on p_9. The remaining twelve are points p_{10} cut out by sextics with a node at p_{10} since all sextics on p_{10} with a simple point at p_{10} touch the cubic C_{10} at p_{10} and not N. Hence in a pencil of sextics with nodes at $p_1, \ldots, p_9$ there are precisely twelve rational sextics. In part this conclusion could be drawn as follows: If p_{10} is the 10-th node of a sextic with nodes at $p_1, \ldots, p_9$ then p_{10} lies both on N and on the 9-ic N'

formed like N with triple points at $p_1, \ldots, p_7, p_9$. Then N and N' meet in 7×9 points at $p_1, \ldots, p_7$ and in 2×3 points at p_8, p_9, whence p_{10} is one of the twelve remaining intersections. Thus there are *at most* twelve positions of p_{10}. It appears therefore that the three conditions that N be on p_9 and p_{10} and that N' be on p_{10} are necessary and sufficient conditions that P_{10}^2 be the nodes of a rational sextic.

The relation between p_9 and p_{10} gives rise to a symmetrical (12, 12) correspondence, T, on N. The valence of T is 3. For if C_8 is a set of the g_8^1, and C_2 the residue of that set on p_9, if K is a canonical set in g_6^3, and G a set of the g_4^1 considered above, and if S_{12} is the set of twelve positions of p_{10} when p_9 is given, then $S_{12}+C_2$ is the set of fourteen double points of the g_4^1. Hence $K+2G\equiv S_{12}+C_2$,* where now the equivalence refers to point groups on N. But $G+2p_9\equiv K$, and $C_2+p_9\equiv C_3$, and $2C_3\equiv K$ whence $S_{12}+3p_9\equiv 2K+C_8$. Hence if p_9' is any other point on N and S_{12}' its set of twelve additional nodes $S_{12}+3p_9\equiv S_{12}'+3p_9'$, or T has the valence $\gamma=3$. Then according to the well-known formula $\alpha+\beta+2p\gamma$, T has $12+12+24=48$ coincidences. These arise from those positions of p_9 where a rational sextic of the web has a tacnode, but also from the twelve points $D_1, \ldots, D_{12}$. For if C_λ has a node at D on N, then C_λ^2 meets N four times at D. Of this $4D$, the set $2D$ is eliminated in forming g_4^1, but $2D$ is left and D is a double point of g_4^1. Thus D belongs to the set S_{12} which corresponds to D in T and is therefore a coincidence. Hence

(14) *There are thirty-six sextics with eight given nodes which have an additional tacnode.*

Thus a sextic with a tacnode has only eight absolute constants. Miss Hudson's theorem that any rational sextic S for which a discriminant condition vanishes can be transformed into a sextic with a tacnode shows that S could have only eight absolute constants. For the tacnodal sextic can be transformed back into S by a series of quadratic involutions each with F-points and one fixed point at nodes of the sextic, and by a subsequent projectivity—a process which can introduce no new absolute constants.

The discriminant conditions furnish irrational invariants of the general sextic S. Symmetric combinations of those which lie within one of the five types of Theorem (10) furnish rational projective invariants of S. Symmetric combinations of the whole set of 496 furnish invariants of S under Cremona transformation of S into S'.

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* Severi, "Lezioni di Geometria Algebrica," p. 160.

§ 3. *The Group $\overline{G}_{10,2}$ of S.*

Since $\overline{G}_{10,2}$ is the group of all Cremona transformations which transform S into itself, the elements of $\overline{G}_{10,2}$ will either leave every point on S unaltered or transform the points of S among themselves according to a transformation on the parameter t of S of the form

$$(15) \qquad t'=\frac{at+b}{ct+d}.$$

The group $\gamma_{10,2}$ of transformations (15) thus induced by $\overline{G}_{10,2}$ upon S will be simply isomorphic with $\overline{G}_{10,2}$ if the group $\overline{\overline{G}}_{10,2}$ of Cremona transformations for which every point of S is fixed is merely the identity. Otherwise $\gamma_{10,2}$ is the factor group of $\overline{\overline{G}}_{10,2}$ under $\overline{G}_{10,2}$.

The $\overline{G}_{10,2}$ is generated by the conjugates of the Bertini involution under $G_{10,2}$. If B is the involution with F-points at the nodes $p_1, \ldots, p_8$ of S, then we have just seen that B leaves the points p_9 and p_{10} unaltered and interchanges the two branches of S at each of these nodes. Hence if t_9, t_9' and t_{10}, t_{10}' are the pairs of nodal parameters, the transformation (15) induced by B interchanges the parameters in each pair and is the involution whose fixed points are the Jacobian of the nodal pairs. These fixed points are cut out on S by the curve N outside of P_{10}^2.

Two f-curves may meet at an F-point, say p_i in P_{10}^2, but ordinarily they pass through p_i with different tangents, *i. e.*, they have at p_i different points in common with the f-curve, $f_0(i)$, which is made up of directions at p_i. We say then they have no *proper* intersection at p_i. Two f-curves may be selected so that they have any number of proper intersections. For as the order of the transformations of $G_{10,2}$ increases, the multiplicity of the f-curves of the transformations at F-points also increases so that the number of proper intersections of these f-curves and $f_0(i)$ increases without limit. Any two f-curves without proper intersections are conjugate under $G_{10,2}$. For the first can be transformed into $f_0(10)$ by an operation of $G_{10,2}$ which at the same time transforms the second into an f-curve on P_9^2; and this finally by an operation of $G_{9,2}$ which $f_0(10)$ unaltered can be transformed into $f_0(9)$. Also since every f-curve has precisely two proper intersections with S we have the theorem:

(16) *The group $\gamma_{10,2}$ of transformations* (15) *on S is generated by a conjugate set of involutions each determined by a pair of fixed points which is the Jacobian of the pairs of proper intersections with S of any two f-curves which have no proper intersections with each other.*

One may show in the same way that if ten f-curves are such that no two have proper intersections at P_{10}^2 they define a Cremona transformation of $G_{10,2}$. In fact the signatures of the f-curves furnish the columns of the matrix of L in (1).

If we transform the involution B by L the f-curves $f_0(9)$ and $f_0(10)$ become $f_{r_9}(1^{\alpha_{19}} \ldots . 10^{\alpha_{109}})$ and $f_{r_{10}}(1^{\alpha_{110}} \ldots . 10^{\alpha_{1010}})$. It merely requires a multiplication of three determinants to form the transform $L^{-1}BL$, and after evident reductions we find that the transformed form has coefficients

$$(17) \quad \begin{cases} m' = 17 + 12(r_9 + r_{10}) + 4r_9 r_{10}, \\ r_i' = s_i' = 6 + 2(r_9 + r_{10}) + 6(\alpha_{i9} + \alpha_{i10}) + 2(r_9\alpha_{i10} + r_{10}\alpha_{i9}), \\ (j \neq i)\, \alpha_{ji}' = 2 + 2(\alpha_{j9} + \alpha_{j10} + \alpha_{i9} + \alpha_{i10}) + 2(\alpha_{i9}\alpha_{j10} + \alpha_{j9}\alpha_{i10}), \\ \alpha_{ii} = 3 + 4(\alpha_{i9} + \alpha_{i10}) + 4\alpha_{i9}\alpha_{i10}. \end{cases}$$

(18) *If* $f_{r_9}(1^{\alpha_{19}} \ldots . 10^{\alpha_{109}})$ *and* $f_{r_{10}}(1^{\alpha_{110}} \ldots . 10^{\alpha_{1010}})$ *are two f-curves without proper intersections the conjugate of the Bertini involution determined as in* (16) *by the two when regarded as an element L of $g_{10,2}$ has the coefficients* (17).

The question as to whether $\bar{\bar{G}}_{10,2}$ contains elements other than the identity is related to the question as to whether the two proper intersections of distinct f-curves with S can coincide. For if $C \neq 1$ is an element of $\bar{\bar{G}}_{10,2}$ and leaves every point of S unaltered, it leaves the two directions of S at p_i unaltered, whence the f-curve which corresponds to p_i under C must pass through p_i with these two directions (and in general others). Thus this f-curve and $f_0(p_i)$ have the same pair of proper intersections with S. I am inclined to think that distinct f-curves meet S in distinct pairs, but have no proof that this is true.

PART II.

THE TEN NODES OF THE SYMMETROID.

§ 4. *The Dilation of a Regular Cremona Group.*

A *regular* Cremona transformation in S_k is by definition (P. S. II, § 4) any product of involutions of the type $y_i'y_i = C_i$ $(i = 1, 2, \ldots, k+1)$ where the products are formed with the $(k+1)$ F-points within a given point set as described in the introduction. The regular group $G_{n,k}$ attached to the point set P_n^k, transforms spreads of order x_0 and multiplicities $x_1, \ldots, x_n$ at P_n^k

according to the group $g_{n,k}$ of linear transformations L with coefficients (P. S. II, § 5 (23))

$$\text{(19)}\qquad \begin{pmatrix} (k-1)\mu+1-\rho_1 & -\rho_2 & \vdots & -\rho_n \\ (k-1)\sigma_1 & -\alpha_{11}-\alpha_{12} & \vdots & -\alpha_{1n} \\ (k-1)\sigma_2 & -\alpha_{21}-\alpha_{22} & \vdots & -\alpha_{2n} \\ \cdots & \cdots & \vdots & \cdots \\ (k-1)\sigma_n & -\alpha_{n1}-\alpha_{n2} & \vdots & -\alpha_{nn} \end{pmatrix}.$$

This group $g_{n,k}$ is generated by the permutation $g_{n!}$ of the n variables and the involution $A_{1,2,\ldots,k+1}$ whose coefficients are (P. S. II, § 5)

$$\text{(20)}\qquad \begin{pmatrix} k & -1 & -1 & \vdots & -1 & 0 & \vdots \\ k-1 & 0 & -1 & \vdots & -1 & 0 & \vdots \\ k-1 & -1 & 0 & \vdots & -1 & 0 & \vdots \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \vdots \\ k-1 & -1 & -1 & \vdots & 0 & 0 & \vdots \\ 0 & 0 & 0 & \vdots & 0 & 1 & \vdots \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \vdots \end{pmatrix}.$$

Suppose then that the general element of $G_{n,k}$ has been obtained by forming a proper sequence Π of the products from $g_{n!}$ and $A_{1,\ldots,k+1}$. Consider a set of $n+l$ points in an S_{k+l}, *i. e.*, a set P_{n+l}^{k+l}. In this space separate a set of l of the points P_{n+l}^{k+l} (call these for the moment the *fixed* F-points) and order the remaining n points of P_{n+l}^{k+l} with respect to the points of P_n^k. Then in S_{k+l} form a product Π' of elements from $g_{(n+l)!}$ and $A_{1,\ldots,l,l+1,\ldots,l+k+1}$ in such a way that the last n points of P_{n+l}^{k+l} are permuted like the n points of P_n^k under $g_{n!}$, the first l remaining fixed. This requires that always in using an element A the first l of its F-points shall fall at the first l points of the set P_{n+l}^{k+l}. We shall then say that the element Π' of $G_{n+l,k+l}$ is the *element* Π *of* $G_{n,k}$ *dilated into* S_{k+l}. The element of $g_{n+l,k+l}$ which corresponds to the element Π' dilated from (19) has coefficients

$$\text{(21)}\qquad \begin{pmatrix} (k+l-1)\mu+1 & -\mu & -\mu & \vdots & -\mu & -\rho_1 & \cdots & -\rho_n \\ (k+l-1)\mu & -\mu+1 & -\mu & \vdots & -\mu & -\rho_1 & \cdots & -\rho_n \\ (k+l-1)\mu & -\mu & -\mu+1 & \vdots & -\mu & -\rho_1 & \cdots & -\rho_n \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ (k+l-1)\mu & -\mu & -\mu & \vdots & -\mu+1 & -\rho_1 & \cdots & -\rho_n \\ (k+l-1)\sigma_1 & -\sigma_1 & -\sigma_1 & \vdots & -\sigma_1 & -\alpha_{11} & \cdots & -\alpha_{1n} \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ (k+l-1)\sigma_n & -\sigma_n & -\sigma_n & \vdots & -\sigma_n & -\alpha_{n1} & \cdots & -\alpha_{nn} \end{pmatrix}.$$

In order to prove this we have only to show that the general element (19) multiplied by $A_{1,\ldots,k+1}$ when dilated according to the rule which is evident in (21) is the same as the dilated element (21) multiplied by $A_{1,\ldots,l+1,\ldots,l+k+1}$. We shall omit the verification which depends merely on determinant multiplication. Hence

(22) *The regular Cremona group attached to a set P_n^k in S_k when dilated into S_{k+l} furnishes a subgroup of the regular Cremona group in S_{k+l} attached to the set P_{n+l}^{k+l} which is simply isomorphic with the original group. The dilated group permutes the S_l's in S_{k+l} upon the l fixed F-points just as the original group permutes the points of S_k.*

In fact if the S_l's be cut by an S_k, which does not cut their common S_{l-1}, the original group appears in this S_k.

The following extension of P. S. II, § 4 (17) is now evident.

(23) *The group $G_{n,k}$ contains subgroups simply isomorphic with $G_{n',k'}$ whenever $n' \lesseqgtr n$ and $k' \lesseqgtr k$.*

We shall have occasion to use the dilations into S_3 of the Bertini involution, and of the Geiser involution in S_2 with triple F-points at $p_2, \ldots, p_8$. The matrices of these dilated transformations are, respectively,

$$(24)\quad \begin{pmatrix} 33 & -16 & -6 & -6 & \ldots & -6 \\ 32 & -15 & -6 & -6 & \ldots & -6 \\ 12 & -6 & 3 & -2 & \ldots & -2 \\ 12 & -6 & -2 & 3 & \ldots & -2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 12 & -6 & -2 & -2 & \ldots & 3 \end{pmatrix}, \quad \begin{pmatrix} 15 & -7 & -3 & -3 & \ldots & -3 \\ 14 & -6 & -3 & -3 & \ldots & -3 \\ 6 & -3 & -2 & -1 & \ldots & -1 \\ 6 & -3 & -1 & -2 & \ldots & -1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 6 & -3 & -1 & -1 & \ldots & -2 \end{pmatrix}.$$

§ 5. *The Transforms of the Symmetroid by Regular Cremona Transformation.*

The symmetroid Σ is the quartic surface obtained by equating to zero a symmetric determinant of order 4 whose elements are linear forms. The ten points at which the first minors all vanish form the set P_{10}^3 of nodes of Σ. The enveloping cone of Σ from one of the nodes breaks up into two cones of the third order which meet in the nine lines to the other nodes. If this property appears at one node of a ten-nodal quartic surface, the surface is a symmetroid.

Let us call a set of eight points in space $p_1, \ldots, p_8$ a *half-period set* if on the elliptic quartic through the eight, the parameters satisfy the condition $u_1 + \ldots + u_8 \equiv \omega/2$, where $\omega/2$ is not the zero half-period when $v_1 + \ldots + v_4 \equiv 0$ is the coplanar condition. Let us further call $8+k$ points a *half-period set* if every set of eight in the set of $8+k$ points is itself a half-period set. Then a further property of Σ is that its set of nodes P_{10}^3 is a half-period set.*

From the property of the enveloping cone there follows:

(25) *If nine nodes of a symmetroid are given, the tenth is uniquely determined.*

(26) *A symmetroid is transformed by regular Cremona transformations with $\rho \gtrless 10$ F-points at P_{10}^3 into a symmetroid Σ' whose nodes $P_{10}^{3\prime}$ are congruent to P_{10}^3.*

For first if $p_1, \ldots, p_9$ are given, the line $\overline{p_1p_{10}}$ is determined as the 9-th base line of a pencil of cubic cones on the eight lines $\overline{p_1p_2}, \ldots, \overline{p_1p_{10}}$. Similarly the line $\overline{p_2p_{10}}$ is determined and thereby also the node p_{10}. Secondly a cubic transformation A_{1234} of the type $x_i'x_i = C_i$ $(i=1, \ldots, 4)$ with F-points at $p_1, \ldots, p_4$ transforms Σ into a quartic surface Σ' with nodes at a congruent set Q_{10}^4. Now A_{1234} is the dilation of a ternary quadratic transformation A_{234} which sends nine base points of a pencil of cubics on $p_2p_3p_4$ into a congruent set with a similar base point property whence A_{1234} has the same effect on the nine base lines through p_1, and Σ' is also a symmetroid. Moreover, any regular transformation of the sort described in (26) is a product of such cubic transformations.

It is our primary purpose to show that Σ can be transformed by such regular transformation into only a finite number of projectively distinct symmetroids, or since

(27) *There is but one symmetroid with given nodes*, that from the set P_{10}^3 of nodes of Σ only a finite number of projectively distinct congruent sets Q_{10}^3 can be derived.

In general there is an infinite number of sets Q_{10}^3 congruent to but projectively distinct from P_{10}^3 (P. S. II (14), (18)), and these arise from P_{10}^3 by the operations of the group $G_{10,3}$. If for the set P_{10}^3 of nodes of Σ this number is finite, there must be a subgroup $\overline{G}_{10,3}$ of $G_{10,3}$, which transforms Σ into itself, of finite index under $G_{10,3}$. We shall find that an important subgroup

* Cayley, *Coll. Math. Pap.*, Vol. VII, p. 304; Vol. VIII, p. 25.

$G(2)$ of $\overline{G}_{10,8}$ is generated by the conjugates under $G_{10,8}$ of two types of involutions, namely, the "Kantor involution"* and the dilated Bertini involution. In $g_{10,8}$ there are the isomorphic subgroups $\overline{g}_{10,8}$ and $g(2)$.

The Kantor involution K is that cut out on elliptic quartic curves on $p_1, \ldots, p_7$ by quartic surfaces with nodes at $p_1, \ldots, p_7$. It has for fixed points the 8-th node of such surfaces; and these are the 8-th base point P of the net of quadrics on $p_1, \ldots, p_7$—an isolated fixed point with fixed directions—and the locus of the point p_8 which forms with $p_1, \ldots, p_7$ a half-period set—the Cayley dianome sextic surface. Hence p_8, p_9, p_{10}, the further nodes of Σ, are fixed points of K, and Σ is unaltered by K. This involution is the analog in space of the Bertini involution in the plane.

In order to show that the dilated Bertini involution also leaves Σ unaltered, two lemmas are useful.

(28) *The dilation from p_1 of the Geiser involution with F-points $p_2, \ldots, p_8$ in S_2 is, in S_3, the transformation* (24) *whose two sets of F-points $p_1, \ldots, p_8$ and $q_1, \ldots, q_8$ are projective only when they are half-period sets. If the two sets are thus restricted and coincide in the identical order, the dilated transformation is involutory.*

For the dilation of this involution is found listed in P. S. II, p. 376, as $C(\nu)\,(\nu=-1)$. It is shown there that $C(-1)C(0)=D_1$ or $C(-1)=D_1C(0)$ where $C(0)$ is the Kantor involution determined by $p_2, \ldots, p_8$. It is clear from the parametric equations of D_1 (*loc. cit.*) that its two sets of F-points are projective if they are half-period sets. In this case p_1 is a fixed point of $C(0)$ and the two sets of F-points of $C(-1)$ are projective. If for $C(-1)$, P_8^3 and Q_8^3 coincide then $p_1, \ldots, p_8$ are ordinary points of $[C(-1)]^2$ and $C(-1)$ is involutory.

(29) *The dilation from p_1 of the Bertini involution with F-points $p_2, \ldots, p_9$ in S_2 is, in S_3, the transformation* (24) *whose two sets of F-points $p_1, \ldots, p_9$; $q_1, \ldots, q_9$ are projective only when these sets are half-period sets. If they are thus restricted and coincide in the identical order, the dilated transformation is involutory.*

For in P. S. II, p. 353, the Bertini involution (E_{17}) is expressed as a product of three Geiser involutions (D_8) and from the projectivity of the two

* The Kantor involution appears first in two papers of S. Kantor, "Theorie der periodischen cubischen Transformationen im R_3," this Journal, Vol. XIX (1897), p. 1; and "Theorie der Transformationen im R_3," *Acta Mathematica*, Vol. XXI (1897), p. 1, both of which deal with regular transformations in S_3. A development of the properties of the involution is given by J. R. Conner, "Correspondences Determined by the Bitangents of a Quartic," this Journal, Vol. XXXVIII (1916), p. 155.

sets of seven F-points of the factors, the projectivity of any two corresponding sets of six points from the two sets of eight F-points of the product was derived. Because of the isomorphism between elements in S_2 and their dilations in S_3 the dilated Bertini involution can be expressed as a similar product of three dilated Geiser involutions. Hence by virtue of (28) we can conclude that the pair $p_1 q_1$ and any six further pairs of F-points are projective when $p_1, \ldots, p_9$ is a half-period set. Hence the two sets of F-points of the dilated Bertini involution are projective when one is a half-period set, and if the two sets coincide the square of the transformation is the identity.

We can now proceed with Σ very much as with the sextic S and will state first the analog of Theorem (4), § 1.

(30) *The group, $G(2)$, of regular transformations in S_3, generated by the conjugates of the Kantor and dilated Bertini involutions under $G_{10,3}$, is an invariant subgroup of $\overline{G}_{10,3}$ which leaves Σ unaltered. The isomorphic group, $g(2)$, is that subgroup of $g_{10,3}$ which is congruent to the identity modulo 2.*

Indeed we have already remarked that K leaves Σ unaltered. There follows directly from (29), (25) and (27) that B has the same property. That the conjugates of K, B under $G_{10,3}$ have this property is proved as for the sextic. In order to prove that the involutions generate $G(2)$ we indicate as before by the symbol $\equiv$ equivalence under $G(2)$.

If P_{10}^3 is the set of nodes of Σ it determines a sequence of f-surfaces, the conjugates of the ∞^2 directions about $p_1, \ldots, p_{10}$ under the operations of $G_{10,3}$. We construct like the Table (5) for the sextic the Table (31), discarding as new types those f-surfaces which are equivalent under $G(2)$ to types found earlier. Non-equivalent new types are starred as they occur.

In order to prove the equivalences listed in the table we shall prove first that the following list is valid.

(12°) $$f_3(1^3 2 5^3 678910) \equiv f_3(1^2 2^3 5678910).$$

(13°) $$f_3(1 2^3 5^2 678910) \equiv f_3(1 2^3 3^2 678910).$$

(14°) $$f_2(1 2^2 3456) \equiv f_2(134567^2).$$

(15°) $$f_4(1^3 2^4 345678910) \equiv f_4(1 2^4 3^3 45678910).$$

We begin with the equivalences obtained from K and B,

$$f_0(i) \equiv f_4(i^3 j_1^2 \ldots j_6^2) \equiv f_6(i^3 j_1^6 j_2^2 \ldots j_8^2),$$

and transform them successively by $A_{ij_1j_2j_3}$, $A_{ij_1j_4j_5}$, $A_{ij_1j_6j_7}$, and $A_{ij_1j_8j_9}$ getting

$$\begin{aligned}
f_1(j_1j_2j_3) &\equiv f_3(i^2j_1j_2j_3j_4^2j_5^2j_6^2) &&\equiv f_5(i^2j_1^5j_2j_3j_4^2\ldots j_8^2),\\
f_2(ij_1^2j_2j_3j_4j_5) &\equiv f_2(ij_2j_3j_4j_5j_6^2) &&\equiv f_4(ij_1^4j_2j_3j_4j_5j_6^2j_7^2j_8^2),\\
f_3(i^2j_1^3j_2\ldots j_7) &\equiv f_3(i^2j_1\ldots j_5j_6^3j_7) &&\equiv f_3(j_1^3j_2\ldots j_7j_8^2),\\
f_4(i^3j_1^4j_2\ldots j_9) &\equiv f_6(i^5j_1^4j_2\ldots j_5j_6^3j_7j_8^3j_9^3) &&\equiv f_4(ij_1^4j_2\ldots j_8j_9^3).
\end{aligned}$$

(31)

Type	operated upon by	becomes	which is	
$f_0(1)$	A_{1234}	$f_1(234)$*		
	A_{2345}	$f_0(1)$		
$f_1(234)$	A_{1234}	$f_0(1)$		
	A_{1235}	$f_1(234)$		
	A_{1256}	$f_2(12^23456)$*		
	A_{1567}	$f_3(1^22345^26^27^2)$	$\equiv f_1(234)$	(1^0)
$f_2(12^23456)$	A_{1234}	$f_1(256)$		
	A_{3456}	$f_2(12^23456)$		
	A_{1237}	$f_2(12^23456)$		
	A_{4567}	$f_3(12^234^25^26^27)$	$\equiv f_1(137)$	(1^0)
	A_{1278}	$f_3(1^22^3345678)$*		
	A_{5678}	$f_4(12^2345^36^37^28^2)$	$\equiv f_2(12^23456)$	(2^0)
	A_{2789}	$f_4(12^434567^28^29^2)$	$\equiv f_2(12^23456)$	(3^0)
	A_{1789}	$f_5(1^42^234567^38^39^3)$	$\equiv f_3(1^234567^389)$	(4^0)
	A_{78910}	$f_6(12^234567^48^49^410^4)$	$\equiv f_2(12^23456)$	(5^0)
$f_3(1^22^3345678)$	A_{1234}	$f_2(12^25678)$		
	A_{2345}	$f_3(1^22^3345678)$		
	A_{1345}	$f_4(1^32^33^24^25^2678)$	$\equiv f_2(123^2678)$	(2^0)
	A_{3456}	$f_5(1^22^33^34^35^36^378)$	$\equiv f_3(1^22^3345678)$	(6^0)
	A_{1239}	$f_3(1^22^3345678)$		
	A_{2789}	$f_4(1^22^434567^28^29)$	$\equiv f_2(1^234569)$	(3^0)
	A_{1789}	$f_5(1^42^334567^38^39^2)$	$\equiv f_3(1^22^3345678)$	(4^0)
	A_{6789}	$f_6(1^22^33456^47^48^49^3)$	$\equiv f_2(1^223459)$	(7^0)
	A_{12910}	$f_4(1^32^43\ldots10)$*		
	A_{23910}	$f_5(1^22^53\ldots78^39^210^2)$	$\equiv f_3(1^22^3345678)$	(8^0)
	A_{13910}	$f_6(1^52^33\ldots78^49^310^3)$	$\equiv f_4(1^32\ldots78^4910)$	(9^0)
	A_{78910}	$f_7(1^22^334567^58^59^410^4)$	$\equiv f_3(1^22^3345678)$	(10^0)
$f_4(1^32^43\ldots.10)$	A_{1234}	$f_3(1^22^35678910)$		
	A_{2345}	$f_5(1^32^53^24^25^2678910)$	$\equiv f_3(12^33^2678910)$	(8^0)
	A_{1345}	$f_6(1^52^43^34^35^3678910)$	$\equiv f_4(1^32^43\ldots10)$	(9^0)
	A_{3456}	$f_8(1^32^43^54^55^56^578910)$	$\equiv f_4(1^32^43\ldots10)$	(11^0)

In these transforms we find (12^0), (13^0), (14^0), (15^0) as well as (1^0), (2^0) and (9^0). Also (2^0) is transformed by A_{1234} into (6^0), whence (6^0) is valid. Since (8^0) is transformed by A_{1234} into (13^0), (8^0) also is valid. Again (7^0) is

transformed by A_{1267} and the use of (13°) into (4°), and (4°) by A_{1789} into (14°). Also (3°) is transformed by A_{1234} into one proved above. The equivalence (11°) is transformed by A_{1234} into (10°), and (10°) by A_{1278} into (5°). Finally, by using (14°) we write (5°) as $f_6(12^2 34567^4 8^4 9^4 10^4) \equiv f_2(134567^2)$ and this is transformed by A_{1789} and the use of (13°) into (4°). According to the equivalences derived above from the conjugates of K and B we find that all f-surfaces whose signatures are congruent modulo 2 are equivalent under $G(2)$ which completes the proof of (30).

The factor group of $g(2)$ under $g_{10,3}$ is the group $g_{10,3}^{(2)}$ of transformations L reduced modulo 2. According to the table (T. M. Groups, p. 337, $\varkappa=3, \nu=2$) this group has the order $\mu=2^9 . 2^{16}(2^8-1)(2^6-1)(2^4-1)(2^2-1)$. Also μ is the index of $G(2)$ under $G_{10,3}$. There may be elements in $G_{10,3}$ other than those in $G(2)$ which leave Σ unaltered. Consider the μ transforms of Σ under $G_{10,3}$. In these transforms we find that the f-surface $f_0(10)$ is transformed into 2^9 conjugates not equivalent under $G(2)$. These are of the five types listed in the first column of Table (31), there being $\binom{10}{1}, \binom{10}{3}, \binom{10}{5}, \binom{10}{7}, \binom{10}{9}$ of the respective types. Hence, under the operations of $G_{10,3}$ for which p_{10} is an ordinary point, we would find only $\mu'=\mu/2^9$ transforms of Σ. Under the latter operations the f-surface $f_0(9)$ is transformed into 2^8-1 conjugates not equivalent under $G(2)$, namely the $\binom{9}{1}, \binom{9}{3}, \binom{9}{5}, \binom{9}{7}$ of the first four types just mentioned. Hence under the operations of $G_{10,3}$ for which both p_9 and p_{10} are ordinary points, we would get only $\mu''=\mu'/(2^8-1)=2^{16}(2^6-1)(2^4-1)(2^2-1)$ transforms of Σ, and these recur in sets of $8!$ obtained by permutation of $p_1, \ldots, p_8$. Thus we should get only $\mu''/8!=2.2^6.36$ projectively distinct sets of nodes $p_1, \ldots, p_8$. On the other hand we have proved (P. S. II, p. 377 (46)) that when P_8^3 is a half-period set, there are only $2^6.36$ projectively distinct sets congruent in some order to P_8^3.

This indicates the existence of Cremona transformations not in $G(2)$ which have their F-points at $p_1, \ldots, p_8$ alone and which transform Σ into itself. Indeed

(32) *The dilated Geiser involution with F-points at the nodes $p_1, \ldots, p_8$ of Σ transforms Σ into itself and interchanges the nodes p_9 and p_{10}.*

For let us first recall with Rohn* that when the first seven nodes of Σ are given, the other three lie on Cayley's dianome sextic surface with triple points at $p_1, \ldots, p_7$. Having chosen p_8 on this surface, the other two nodes lie on

*K. Rohn, "Die Flächen vierter Ordnung," etc., Jablonowski'schen Preisschrift, Leipzig (1886), §§ 9, 10, 11.

Cayley's dianodal curve of order 18 with planar triple points at the eight nodes. As Rohn remarks, the ninth being chosen, the tenth is uniquely determined if the quartic is to be a symmetroid. This follows immediately from (25). Thus there is on the dianodal curve an involution of pairs of nodes of symmetroids. Now this involution is effected by the Geiser involution dilated from p_1 (and therefore also that the Geiser involution dilated from any other of the eight nodes). For since the eight nodes are a half-period set, the dilated transformation is involutory (28) when its two sets of eight F-points coincide. Moreover, the dilated transformation is regular and transforms symmetroids into symmetroids (26) and therefore leaves the dianodal curve unaltered. If p_9, p_9' are a copair of the dilated involution on the curve, then from (22) the lines $\overline{p_1p_9}$, $\overline{p_1p_9'}$ form with $\overline{p_1p_2}, \ldots, \overline{p_1p_8}$ the base lines of a pencil of cubic cones. But this property is shared by the lines $\overline{p_1p_9}$ and $\overline{p_1p_{10}}$ when p_9, p_{10} are nodes of the same symmetroid. Hence p_9' is p_{10} and the theorem is proved.

Consider now the reduced group $g^{(2)}_{10,8}$ of $g_{10,8}$. The dilated Geiser involution reduced modulo 2 is

$$I_{12\ldots 8}I_{910} \text{ or } x_j'=x_j+(x_1+\ldots+x_8) \quad (j=0,1,\ldots,8), \quad x_9'=x_{10}, \quad x_{10}'=x_9$$

in the notation of T. M. Groups.* This is an element T (cf. p. 326, *loc. cit.*) which lies in the invariant g_{2^9} of $g^{(2)}_{10,8}$. If an element of $G_{10,8}$ leaves Σ unaltered, its conjugates have the same property whence those elements of $g^{(2)}_{10,8}$ conjugate to T under $g^{(2)}_{10,8}$, also correspond to elements of $G_{10,8}$ which leave Σ unaltered. Now the factor group of g_{2^9} under $g^{(2)}_{10,8}$ is the *simple* group $G_{NO}(p=4)$ of the odd and even thetas for $p=4$. Hence there are no further elements of $G_{10,8}$ which leave Σ unaltered since any such element reduced modulo 2 would furnish an invariant subgroup of $g^{(2)}_{10,8}$ larger than g_{2^9} whose factor group under $g^{(2)}_{10,8}$ would be the factor group under G_{NO} of an invariant subgroup of G_{NO} greater than the identity. But no such subgroup of G_{NO} exists. Hence the number $\bar{\mu}$ of transforms of Σ under $G_{10,8}$ is the order of G_{NO}, *i. e.*, $\bar{\mu}=2^{16}(2^8-1)(2^6-1)(2^4-1)(2^2-1)$ and allowing for the permutations of the nodes there are only $\bar{\mu}/10!=2^8.51$ projectively distinct Σ's. Hence

(33) *Under regular Cremona transformation a symmetroid Σ can be transformed into precisely $2^8.51$ projectively distinct Σ''s. The subgroup $\overline{G}_{10,8}$ of $G_{10,8}$, which leaves Σ unaltered is generated by the conjugates*

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* Cf. particularly the table, p. 337, for $\kappa=3$, $\nu=2$, and also (28) and (29) with references there given.

under $G_{10,8}$ of the dilated Geiser involution and the Kantor involution. The corresponding elements of $\bar{g}_{10,8}$ are characterized arithmetically by the fact that when reduced modulo 2 they yield either the identity or elements which transform the forms b_2, b_4 each into itself or into its paired form.† The invariant subgroup $G(2)$ of $G_{10,8}$ for which $g(2)$ is congruent to the identity modulo 2 is generated by the conjugates of the Kantor and dilated Bertini involutions, and has for factor group under $\bar{G}_{10,8}$ an abelian group of involutions of order 2^9. Under $G_{10,8}$ the conjugates of Σ are permuted according to the group of odd and even thetas for $p=4$, the particular types corresponding to the base configurations.‡*

We may note finally the behavior of the discriminant factors of the set P_{10}^8 of nodes of Σ. Due to the equivaleuces under $G(2)$ listed above we find that all of the discriminant conditions are equivalent to the following sets: $\binom{10}{2}$ of type $f_0(i_1i_2)$, $\binom{10}{4}$ of type $f_1(i_1i_2i_3i_4)$, $\binom{10}{8}$ of type $f_2(i_1^2i_2\ldots i_7)$, and $\binom{10}{8}$ of type $f_3(i_1^3i_2^2i_3\ldots i_9)$, or $2(2^8-1)$ in all. But due to the equivalences under elements of $\bar{G}_{10,8}$ not in $G(2)$, these are paired into 2^8-1 pairs, $\binom{10}{2}$ of type $f_0(i_1, i_2)$, $f_3(i_1^2i_3^2i_4\ldots i_{10})$ and $\binom{10}{4}$ of type $f_1(i_1i_2i_3i_4)$, $f_2(i_1^2i_5\ldots i_{10})$. These two types of equivalence lead to the theorems

(34) *If two nodes of a symmetroid coincide, the cubic cone with vertex at any any one of the remaining nodes and on the ten nodes has a double generator on the double node.*

(35) *If four nodes of a symmetroid are in a plane there is a quadric cone with vertex at any one of the four nodes and on the remaining six nodes.*

When none of the discriminant conditions are satisfied they become irrational invariants of the symmetroid whose behavior under $G_{10,8}$ can be described thus:

(36) *Under regular Cremona transformation the 2^8-1 independent discriminant invariants of Σ are permuted like the points of an $S_{2p-1}(p=4)$ under the group of a null system in S_{2p-1}.*

This striking analogy with the 2^6-1 discriminant invariants of P_7^2 (or the ternary quartic for $p=3$; cf. P. S. II, § 8) is undoubtedly significant.

URBANA, ILLINOIS, *May* 15, 1919.

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* The dilated Bertini involution can be generated by dilated Geiser involutions.

† The forms b_2 are $x_{i_1}+x_{i_2}$, $x_{i_1}+\ldots+x_{i_6}$, the forms b_4 are $x_{i_1}+\ldots+x_{i_4}$ and $x_{i_1}+\ldots+x_{i_8}$; paired forms taken together make up $x_{i_1}+\ldots+x_{i_{10}}$ $(i_j=1,\ldots,10)$.

‡ For these configurations cf. a paper of the author on "The Finite Geometry of the Theta Functions," *Trans. Amer. Math. Soc.*, Vol. XIV (1913), p. 271.

Functions of Matrices.

By H. B. Phillips.

1. It is the purpose of the present paper to study the functions represented by polynomials or convergent series in a matrix or a finite number of matrices. As the work is concerned mainly with the roots of the matrices, the fundamental facts about the roots are first briefly developed.*

By a matrix of the n-th order is meant a square array of n^2 elements a_{ik}, $i, k=1, 1, \ldots, n$,

$$A=\begin{Vmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots & \ldots \\ a_{n1} & a_{n2} \ldots a_{nn} \end{Vmatrix}=\|a_{ik}\|.$$

The matrices considered in this paper will all be of the same order.

The determinant

$$|A|=\begin{vmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots & \ldots \\ a_{n1} & a_{n2} \ldots a_{nn} \end{vmatrix}=|a_{ik}|$$

is called the determinant of A. When this determinant is zero, A is called singular.

The sum of two matrices $A=\|a_{ik}\|$ and $B=\|b_{ik}\|$ is the matrix

$$A+B=\|a_{ik}+b_{ik}\|.$$

More generally, if λ and μ are numbers, $\lambda A+\mu B=\|\lambda a_{ik}+\mu b_{ik}\|$. A matrix is zero when and only when all its elements are zero.

The product AB of $A=\|a_{ik}\|$ and $B=\|b_{ik}\|$ is the matrix

$$AB=\|\alpha_{ik}\|, \quad \text{where} \quad \alpha_{ik}=\sum_{j=1}^{n} a_{ij}b_{jk}.$$

* The general theory of matrices is given in Bocher's "Introduction to Higher Algebra." A very complete bibliography of literature before 1907 is given in James Byrnie Shaw's "Synopsis of Linear Associative Algebra," published by the Carnegie Institution of Washington.

The products AB and BA are not in general equal. If these products are equal, A and B are called commutative.

The products of three or more matrices are associative, that is,

$$(AB)C=A(BC)=ABC.$$

The determinant of a product of matrices is equal to the product of their determinants, for example, $|ABC|=|A|\cdot|B|\cdot|C|$.

The matrix

$$I=\left\|\begin{matrix} 1 & 0 & 0 \ldots\ldots 0 \\ 0 & 1 & 0 \ldots\ldots 0 \\ \ldots & \ldots & \ldots\ldots\ldots \\ 0 & 0 & 0 \ldots\ldots 1 \end{matrix}\right\|,$$

in which the elements a_{ii} are all unity and the others zero, is called the unit matrix. It is easy to see that $AI=IA=A$.

If the determinant of A is not zero, there is a matrix A^{-1} called the reciprocal of A, such that $AA^{-1}=A^{-1}A=I$. In this case, if $AB=CD$, $B=A^{-1}CD$, *i. e.*, we can divide by a non-singular matrix.

It is to be noticed that in both multiplication and division the operation can be performed on the right or on the left. B is multiplied on the left by A if the result is AB, and on the right by A if the result is BA. Similarly, B is divided on the left by A if the result is $A^{-1}B$ and on the right if the result is BA^{-1}.

2. *Identical Equations.*—The matrices $A, B, \ldots, P$ satisfy the equation

$$A+B+\ldots+P=0$$

if their sum is a matrix $\|\sigma_{ik}\|$ with elements σ_{ik} all zero. Cayley first observed that a matrix of the n-th order satisfies an algebraic equation of the n-th degree. This may be considered a consequence of the following theorem:

THEOREM I. *Let $A=\|a_{ik}\|$, $B=\|b_{ik}\|$, $\ldots$, $P=\|p_{ik}\|$ be a finite number of matrices of the n-th order and let $\sigma_{ik}=\lambda a_{ik}+\mu b_{ik}+\ldots+\rho p_{ik}$, $\lambda, \mu, \ldots, \rho$ being numerical parameters. If $A', B', \ldots, P'$ are commutative with each other and satisfy the equation*

$$AA'+BB'+\ldots+PP'=0, \tag{1}$$

they will also satisfy the n-th degree equation

$$|a_{ik}A'+b_{ik}B'+\ldots+p_{ik}P'|=0, \tag{2}$$

obtained by replacing $\lambda, \mu, \ldots, \rho$ in $|\sigma_{ik}|=0$ by the matrices $A', B', \ldots, P'$, respectively.

This theorem enables us to replace a homogeneous linear equation with matrix coefficients by one of the n-th degree with numerical coefficients. The latter equation will usually have solutions, however, that do not satisfy the former.

To prove the theorem, let E_{ik} be the matrix of the n-th order with all its elements zero except that in the i-th row and k-th column, which is unity. It is readily seen that

$$E_{ij}E_{jk}=E_{ik}, \quad E_{ij}E_{lk}=0, \quad l\neq j. \tag{3}$$

The matrix A can be written

$$A=\sum_{i,\,k=1}^{n} a_{ik}E_{ik},$$

and so (1) is equivalent to

$$\sum_{i,\,k=1}^{n} E_{ik}(a_{ik}A'+b_{ik}B'+\ldots\ldots+p_{ik}P')=0.$$

Multiplying this equation on the left by $E_{11}, E_{12}, \ldots\ldots, E_{1n}$, respectively, and using (3), we get

$$\begin{aligned}
&E_{11}(a_{11}A'+\ldots\ldots+p_{11}P')+E_{12}(a_{12}A'+\ldots\ldots+p_{12}P')+\ldots\ldots+E_{1n}(a_{1n}A'+\ldots\ldots+p_{1n}P')=0,\\
&E_{11}(a_{21}A'+\ldots\ldots+p_{21}P')+E_{12}(a_{22}A'+\ldots\ldots+p_{22}P')+\ldots\ldots+E_{1n}(a_{2n}A'+\ldots\ldots+p_{2n}P')=0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&E_{11}(a_{n1}A'+\ldots\ldots+P_{n1}P')+E_{12}(a_{n2}A'+\ldots\ldots+p_{n2}P')+\ldots\ldots+E_{1n}(a_{nn}A'+\ldots\ldots+p_{nn}P')=0.
\end{aligned}$$

Since $A', B', \ldots\ldots, P'$ are commutative, we can eliminate $E_{12}, E_{13}, \ldots\ldots, E_{1n}$ by multiplying these equations (on the right) by the cofactors of the corresponding elements in the first column of the determinant

$$|a_{ik}A'+b_{ik}B'+\ldots\ldots+p_{ik}P'| \tag{4}$$

and adding. The result is

$$E_{11}|a_{ik}A'+b_{ik}B'+\ldots\ldots+p_{ik}P'|=0. \tag{5}$$

If we expand (4) and combine terms, the result is a matrix. Equation (5) shows that all the elements in the first row of that matrix are zero. For, the product of E_{11} and any matrix $\|\alpha_{ik}\|$ has as first row $\alpha_{11}, \alpha_{12}, \ldots\ldots, \alpha_{1n}$. If then $E_{11}\|\alpha_{ik}\|$ is zero, the elements $\alpha_{11}, \alpha_{12}, \ldots\ldots, \alpha_{1n}$ must all be zero. Similarly, we could eliminate $E_{11}, E_{13}, \ldots\ldots, E_{1n}$ and so get

$$E_{12}|a_{ik}A'+b_{ik}B'+\ldots\ldots+p_{ik}P'|=0,$$

showing that all the elements in the second row of the matrix represented by (4) are zero. By continuation of this argument we conclude finally that

$$|a_{ik}A'+b_{ik}B'+\ldots\ldots+p_{ik}P'|=0.$$

3. *Characteristic Equation.*—Let I be the unit matrix. Applied to the identity $AI-IA=0$, Theorem I gives

$$\begin{vmatrix} a_{11}I-A, & a_{12}I, & \dots & a_{1n}I \\ a_{21}I, & a_{22}I-A, & \dots & a_{2n}I \\ \dots & \dots & \dots & \dots \\ a_{n1}I, & a_{n2}I, & \dots & a_{nn}I-A \end{vmatrix}=0.$$

This is an equation of the n-th degree in A called the *characteristic* equation of A. Arranged in descending powers of A (with a change of sign if n is odd), it takes the form

$$\phi(A)=A^n+\alpha_1A^{n-1}+\dots+\alpha_nI=0. \tag{6}$$

Let $a_1, a_2, \dots, a_n$ be the roots of the equation

$$\phi(r)=r^n+\alpha_1r^{n-1}+\dots+\alpha_n=0.$$

Then $\phi(r)$ can be factored in the form

$$\phi(r)=(r-a_1)(r-a_2)\dots(r-a_n).$$

Since this is an identity in r, the coefficients of each power of r on the two sides of the equation are equal. It will then still hold when r is replaced by A, and $a_1, \dots, a_n$ by $a_1I, \dots, a_nI$. Hence

$$\phi(A)=(A-a_1I)(A-a_2I)\dots(A-a_nI).$$

The numbers $a_1, a_2, \dots, a_n$ are called roots of the matrix A. These roots satisfy the equation

$$\phi(r)=\begin{vmatrix} a_{11}-r & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-r & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-r \end{vmatrix}=0.$$

This expresses that the determinant $|A-rI|$ is zero. The roots of A are therefore the values of r for which the determinant $|A-rI|$ is zero.

A matrix of the n-th order may satisfy an equation of lower than the n-th degree. The equation of lowest degree satisfied by a given matrix will, however, be unique. For, if A satisfies two equations of the m-th degree, we can eliminate A^m from them and so find an equation of lower degree satisfied by A. Let

$$\psi(A)=A^m+c_1A^{m-1}+\dots+c_mI=0$$

be the equation of lowest degree satisfied by A. This is sometimes called the reduced equation for A. It is clear that

$$\psi(r)=r^m+c_1r^{m-1}+\dots+c_m$$

is a factor of $\phi(r)$. For, by division, we get

$$\phi(r)=Q(r)\psi(r)+R(r),$$

where $Q(r)$ is the quotient, and $R(r)$, of degree less than m, is the remainder. Since this is an identity in r, we can replace r by A and so get

$$\phi(A)=Q(A)\psi(A)+R(A).$$

But $\phi(A)$ and $\psi(A)$ are both zero. If then $R(r)$ were not identically zero, $R(A)=0$ would be an equation of lower than the m-th degree satisfied by A. Hence $R(r)=0$ and $\phi(r)=Q(r)\psi(r)$.

This shows that all the roots of the reduced equation are roots of the characteristic equation. Conversely, all the roots of the characteristic equation satisfy the reduced equation. For, since $\psi(r)$ is a polynomial, we can factor $\psi(r)-\psi(s)$ in the form

$$\psi(r)-\psi(s)=(r-s)P(r,s),$$

where $P(r,s)$ is a polynomial in r and s. Since this is an identity we can replace s by A and r by rI. Then, since $\psi(A)=0$,

$$\psi(r)I=(rI-A)P(rI,A).$$

Equating determinants of the two sides, we get

$$[\psi(r)]^n=|rI-A|\cdot|P(rI,A)|.$$

If now r is a root of A, $|rI-A|=0$, and so $\psi(r)=0$ which was to be proved.

4. *Associated Roots.*—THEOREM II. *If $A, B, \ldots, P$ are commutative matrices with roots $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$, etc., those roots can be arranged in sets $a_i, b_i, \ldots, p_i$, $i=1, 2, \ldots, n$ such that, if $f(a, b, \ldots, p)$ is any polynomial in $a, b, \ldots, p$, the roots of the matrix $f(A, B, \ldots, P)$ are*

$$f(a_i, b_i, \ldots, p_i), \quad i=1, 2, \ldots, n.$$

This theorem is due to Frobenius.* The following proof is somewhat more direct than the one given by him. Let $f_1(A, B, \ldots, P)$, $f_2(A, B, \ldots, P)$, etc., be polynomials and

$$f(A, B, \ldots, P)=\sum_{i=1}^{k}\lambda_i f_i(A, B, \ldots, P), \tag{7}$$

$\lambda_1, \lambda_2, \ldots, \lambda_k$ being arbitrary numbers. Since $A, B, \ldots, P$ have characteristic equations of the n-th degree, there are only a finite number of linearly independent polynomials in those matrices. We may consider $f_i(A, B, \ldots, P)$, $i=1, 2, \ldots, k$ as those polynomials, and so by a proper choice of $\lambda_1, \lambda_2, \ldots, \lambda_k$ make $f(A, B, \ldots, P)$ equal to any given polynomial in $A, B, \ldots, P$.

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* Sitzungsberichte Berliner Akademie (1896), p. 602.

By Taylor's theorem,

$$f(A, B, \ldots, P) - f(a_i, b_j, \ldots, p_k)I$$
$$= (A - a_i I)L + (B - b_j I)M + \ldots + (P - p_k I)R, \quad (8)$$

where $L, M, \ldots, R$ are polynomials in $A, B, \ldots, P, a_i, b_j$, etc. Form the product

$$\Pi[f(A, B, \ldots, P) - f(a_i, b_j, \ldots, p_k)I]$$

of all the differences (8) obtained by letting $i, j, \ldots, k$ vary independently from 1 to n. This product is zero. For when the product is expanded in terms of the quantities $(A - a_i I)$, $(B - b_j I)$, etc., each term of the result not containing $(B - b_j I), \ldots, (P - p_k I)$, $j, \ldots, k$ being definitely assigned, will contain all the factors $(A - a_i I)$, $i = 1, 2, \ldots, n$. Hence each term contains one or more of the products

$$\begin{gathered}(A - a_1 I)(A - a_2 I) \ldots (A - a_n I), \\ (B - b_1 I)(B - b_2 I) \ldots (B - b_n I), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ (P - p_1 I)(P - p_2 I) \ldots (P - p_n I).\end{gathered}$$

These all vanish because the result of equating any one of them to zero is the characteristic equation of one of the matrices. Therefore

$$\Pi[f(A, B, \ldots, P) - f(a_i, b_j, \ldots, p_k)I] = 0 \quad (9)$$

for all values of the parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$.

Equation (9) is a polynomial equation in $f(A, B, \ldots, P)$. If the equation of least degree satisfied for all values of $\lambda_1, \lambda_2, \ldots, \lambda_k$ by that matrix is

$$\phi(x) = 0, \quad (10)$$

$\phi(x)$ will be a divisor of

$$\Pi[x - f(a_i, b_j, \ldots, p_k)].$$

The roots of (10), *i. e.*, the roots of $f(A, B, \ldots, P)$, are all therefore obtained by assigning proper values to $i, j, \ldots, k$ in

$$f(a_i, b_j, \ldots, p_k). \quad (11)$$

We can therefore find n sets of constants $\alpha_i, \beta_i, \ldots, \rho_i$ (namely, the sets of values $a_i, b_j, \ldots, p_k$ in (11)) such that the roots of $f(A, B, \ldots, P)$ are

$$f(\alpha_i, \beta_i, \ldots, \rho_i), \quad i = 1, 2, \ldots, n. \quad (12)$$

In particular, if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are so chosen that $f(A, B, \ldots, P) = A$, we shall have $f(\alpha_i, \beta_i, \ldots, \rho_i) = \alpha_i$. The roots of A are therefore $\alpha_1, \alpha_2, \ldots, \alpha_n$. Similarly, the roots of B are $\beta_1, \beta_2, \ldots, \beta_n$, etc. By a change of notation we can then make $\alpha_i = a_i$, $\beta_i = b_i$, etc. The roots of $f(A, B, \ldots, P)$ are then $f(a_i, b_i, \ldots, p_i)$, which was to be proved.

5. *Canonical Form.*—Two sets of associated roots $a_1, b_1, \ldots, p_1$ and $a_2, b_2, \ldots, p_2$ will be considered different unless

$$a_1 = a_2,\ b_1 = b_2,\ \ldots,\ p_1 = p_2.$$

Let
$$a_i,\ b_i,\ \ldots,\ p_i, \quad i = 1, 2, \ldots, m$$

be the different sets of roots of the commutative matrices $A, B, \ldots, P$. Let $\lambda, \mu, \ldots, \rho$ be numbers not satisfying any of the equations

$$\lambda a_i + \mu b_i + \ldots + \rho p_i = \lambda a_k + \mu b_k + \ldots + \rho p_k, \quad i \neq k.$$

By Theorem II, the distinct roots of $\lambda A + \mu B + \ldots + \rho P$ are

$$\lambda a_i + \mu b_i + \ldots + \rho p_i, \quad i = 1, 2, \ldots, m.$$

The factors of its characteristic equation are

$$\lambda A + \mu B + \ldots + \rho P - (\lambda a_i + \mu b_i + \ldots + \rho p_i) I$$
$$= \lambda (A - a_i I) + \mu (B - b_i I) + \ldots + \rho (P - p_i I),$$

and the equation of least degree satisfied for all values of $\lambda, \mu, \ldots, \rho$ is

$$\prod_{i=1}^{m} [\lambda (A - a_i I) + \mu (B - b_i I) + \ldots + \rho (P - p_i I]^{r_i} = 0, \tag{13}$$

the numbers $r_1, r_2, \ldots, r_m$ being the multiplicities of the roots.

Let $\psi_1 = \prod_{i=2}^{m} [\lambda (A - a_i I) + \mu (B - b_i I) + \ldots + \rho (P - p_i I)]^{r_i}$

and, generally, let ψ_j be the product of all the factors in (13) except

$$[\lambda (A - a_j I) + \mu (B - b_j I) + \ldots + \rho (P - p_j I)]^{r_j}.$$

It is clear that

$$\psi_i \psi_k = 0, \quad i \neq k, \tag{14}$$

for the product contains all the factors in the left member of (13). Let ψ_i' be the function obtained by replacing $\lambda, \mu, \ldots, \rho$ in ψ_i by $\lambda', \mu', \ldots, \rho'$. Consider the product

$$[\lambda' (A - a_1 I) + \ldots + \rho' (P - p_1 I)]^{r_1} \psi_1. \tag{15}$$

When this is multiplied by ψ_1' the result is zero because

$$[\lambda' (A - a_1 I) + \ldots + \rho' (P - p_1 I)]^{r_1} \psi_1' = 0.$$

Also the product of (15) and any one of the functions $\psi_2, \psi_3, \ldots, \psi_n$ is zero by (14). Hence

$$[\lambda' (A - a_1 I) + \ldots + \rho' (P - p_1 I)]^{r_1} \psi_1 (\psi_1' + \psi_2 + \ldots + \psi_m) = 0. \tag{16}$$

The roots of the matrix

$$\psi_1' + \psi_2 + \ldots + \psi_m \tag{17}$$

are obtained by replacing $A, B, \ldots, P$ by $a_i, b_i, \ldots, p_i$. When they are replaced by $a_1, b_1, \ldots, p_1$ the result is not zero because ψ_1' is not zero, whereas

$$\psi_2=\psi_3=\ldots=\psi_m=0.$$

In a similar way it is seen that the other roots of (17) are not zero. We can therefore divide (16) by (17) and so get

$$[\lambda'(A-a_1I)+\ldots+\rho'(P-p_1I)]^{r_1}\psi_1=0. \tag{18}$$

Since $\lambda', \mu', \ldots, \rho'$ are arbitrary, if (18) is expanded in powers of those parameters, each coefficient will vanish. Hence

$$(A-a_1I)^{\alpha_1}(B-b_1I)^{\beta_1}\ldots(P-p_1I)^{\rho_1}\psi_1=0$$

for all positive integral values of $\alpha_1, \beta_1, \ldots, \rho_1$ such that

$$\alpha_1+\beta_1+\ldots+\rho_1=r_1.$$

In a similar way it is shown that

$$(A-a_iI)^{\alpha_i}(B-b_iI)^{\beta_i}\ldots(P-p_iI)^{\rho_i}\psi_i=0, \tag{19}$$

for all positive integral exponents such that

$$\alpha_i+\beta_i+\ldots+\rho_i=r_i. \tag{20}$$

Let
$$\psi=\psi_1+\psi_2+\ldots+\psi_m. \tag{21}$$

As in case of (17) it is shown that none of the roots of (21) are zero. We can then define a set of functions ϕ_i by the equations

$$\phi_i=\frac{\psi_i}{\psi}, \quad i=1, 2, \ldots, m. \tag{22}$$

It is clear that

$$\phi_1+\phi_2+\ldots+\phi_m=I. \tag{23}$$

Also, by (14),

$$\phi_i\phi_k=0, \quad i\neq k. \tag{24}$$

Multiplying (23) by ϕ_i we therefore get

$$\phi_i^2=\phi_i, \quad i=1, 2, \ldots, m. \tag{25}$$

Let $f(a, b, \ldots, p)$ be a polynomial in $a, b, \ldots, p$ and

$$f_{\alpha,\beta,\ldots,\rho}(a, b, \ldots, p)=\frac{\partial^{\alpha+\beta+\ldots+\rho}}{\partial a^\alpha\partial b^\beta\ldots\partial p^\rho}f(a, b, \ldots, p).$$

By Taylor's theorem

$$f(a, b, \ldots, p)=f(a_i, b_i, \ldots, p_i) \\ +\sum_\alpha\sum_\beta\ldots\sum_\rho f_{\alpha,\beta,\ldots,\rho}(a_i, b_i, \ldots, p_i)\frac{(a-a_i)^\alpha}{\lfloor\alpha}\ldots\frac{(p-p_i)^\rho}{\lfloor\rho}.$$

Since this is an identity in the variables $a, b, \ldots, p$, we can replace them by $A, B, \ldots, P$. Then

$$f(A, \ldots, P) = f(a_i, \ldots, p_i)I + \sum_{\alpha} \ldots \sum_{\rho} f_{\alpha, \ldots, \rho}(a_i, \ldots, p_i) \frac{(A-a_iI)^{\alpha}}{\underline{|\alpha}} \cdots \frac{(P-p_iI)^{\rho}}{\underline{|\rho}}. \quad (26)$$

Let $\qquad A_i = \phi_i(A-a_iI), \ldots, P_i = \phi_i(P-p_iI).$

From (25) it follows that

$$A_i^{\alpha} B_i^{\beta} \ldots P_i^{\rho} = \phi_i (A-a_iI)^{\alpha}(B-b_iI)^{\beta} \ldots (P-p_iI)^{\rho}.$$

Equations (19), (20), and (22) show that this is zero if $\alpha+\beta+\ldots+\rho \geqq r_i$. Finally, if we multiply (26) by ϕ_i, sum for $i=1, 2, \ldots, m$, and use (23) we get

$$f(A, \ldots, P) = \sum_{i=1}^{m} \phi_i f(a_i, \ldots, p_i) + \sum_{i=1}^{m} \sum_{\alpha} \ldots \sum_{\rho} f_{\alpha, \ldots, \rho}(a_i \ldots p_i) \frac{A_i^{\alpha} \ldots P_i^{\rho}}{\underline{|\alpha} \ldots \underline{|\rho}}, \quad (27)$$

the summation including powers $A_i^{\alpha} \ldots P_i^{\rho}$ for which

$$\alpha+\beta+\ldots+\rho < r_i. \quad (28)$$

Equation (27) gives a form in which any polynomial in the given commutative matrices $A, B, \ldots, P$ can be expressed. We shall refer to it as the canonical form for a function of the matrices.* Its most important property is expressed in the following theorem:

THEOREM III. *If $A, B, \ldots, P$ are commutative matrices with corresponding roots $a_i, b_i, \ldots, p_i$ and $f(a, b, \ldots, p)$ is any polynomial, the matrix $f(A, B, \ldots, P)$ can be expressed as a linear function of matrices depending only on $A, B, \ldots, P$, the coefficients in the linear functions being obtained by substituting each set of roots $a_i, b_i, \ldots, p_i$ in $f(a, b, \ldots, p)$ and in its partial derivatives of order lower than the multiplicity of the root $\lambda a_i + \mu b_i + \ldots + \rho p_i$ in the equation of least degree satisfied for all values of $\lambda, \mu, \ldots, \rho$ by $\lambda A + \mu B + \ldots + \rho P$.*

A case of particular interest is that of a polynomial such that all the coefficients $f(a_i, \ldots, p_i)$, $f_{\alpha, \ldots, \rho}(a_i, \ldots, p_i)$ in (27) vanish. Then, evidently, $f(A, B, \ldots, P) = 0$. We can consider the values $a_i, b_i, \ldots, p_i$ as coordinates of a point in hyperspace. In the case considered, the function $f(a, b, \ldots, p)$ has a zero of order r_i at the point $(a_i, b_i, \ldots, p_i)$. Therefore we have proved

* The formula for a function of a single matrix with distinct roots was given by Sylvester, *Comptes Rendus*, Vol. XCIV (1882), p. 55. The case of a single matrix with repeated roots was given by A. Buchheim, *Phil. Mag.*, (5) 22 (1886), pp. 173–174.

THEOREM IV. *If the polynomial $f(a, b, \ldots, p)$ has at each of the points $(a_i, b_i, \ldots, p_i)$, $i=1, 2, \ldots, m$, a zero of order equal to or greater than the multiplicity of the corresponding root in the equation of least degree satisfied for all values of $\lambda, \mu, \ldots, \rho$ by $\lambda A+\mu B+\ldots+\rho P$ then $f(A, B, \ldots, P)=0$.*

If the matrices ϕ_i, $A_i^a B_i^\beta \ldots P_i^\rho$ in (27) are linearly independent, Theorem IV expresses the necessary and sufficient condition that $f(A, B, \ldots, P)$ vanish. This is easily shown to be true for polynomials in a single matrix.* For polynomials in two or more matrices such may not be the case. For instance, A and B could be equal. Then A_i and B_i would be equal.

6. *Commutative Matrices not Expressible as Polynomials in the same Matrix.*—The simplest illustration of commutative matrices is furnished by polynomials in a single matrix. If

$$A=\alpha_1\phi^p+\alpha_2\phi^{p-1}+\ldots+\alpha_p I, \quad B=\beta_1\phi^q+\beta_2\phi^{q-1}+\ldots+\beta_i I,$$

obviously A and B are commutative. If it were true that any two commutative matrices could be so expressed,† by a repetition of the process, any finite number could be expressed as polynomials in the same matrix. The results of the preceding sections could then be more readily obtained by using these expressions. That such is not the case will now be shown by a simple example. Let

$$A=\begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad B=\begin{Vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}.$$

By direct multiplication it is shown that

$$A^2=B^2=AB=BA=0. \tag{29}$$

Hence A and B are commutative. Suppose they are expressible as polynomials in ϕ. Since the characteristic equation for ϕ is of third degree, the expressions can be reduced to the form

$$A=\alpha_1\phi^2+\alpha_2\phi+\alpha_3 I, \quad B=\beta_1\phi^2+\beta_2\phi+\beta_3 I. \tag{30}$$

If α_1 and β_1 are both zero, these polynomials are of first degree. If not, we form the expression

$$(B_1A-\alpha_1B)^2=[B_1(\alpha_2\phi+\alpha_3 I)-\alpha_1(\beta_2\phi+\beta_3 I)]^2. \tag{31}$$

Equation (29) shows that this is zero. The expression

$$[\beta_1(\alpha_2 x+\alpha_3)-\alpha_1(\beta_2 x+\beta_3)]^2$$

* See Metzler, AMERICAN JOURNAL, Vol. XIV (1892), p. 339.

† Frobenius raised this question in the article to which we have previously referred, but stated that he had not decided whether it could be done or not.

is not identically zero, however, because A and B do not satisfy an equation of the form $\beta_1 A - \alpha_1 B = 0$. By the use of (31) we can then reduce (30) to the form

$$A = a_1\phi + a_2 I, \quad B = b_1\phi + b_2 I. \tag{32}$$

The coefficients a_1 and b_1 can not be zero because A and B are not multiples of the unit matrix. Hence, as before, we form the expression

$$(b_1 A - a_1 B)^2 = (b_1 a_2 - a_1 b_2)^2 I = 0.$$

This shows that $b_1 a_2 - a_1 b_2 = 0$, and so, from (32), $b_1 A - a_1 B = 0$. Since A is not a multiple of B this is impossible. Therefore A and B cannot be expressed as polynomials in ϕ.

7. *Limits and Convergence.*—A variable matrix $\phi = \|r_{ik}\|$ is said to approach a matrix $A = \|a_{ik}\|$ as limit if

$$\operatorname{Lim} r_{ik} = a_{ik}, \quad i, k = 1, 2, \ldots, n.$$

In this case it is clear that each root of ϕ will approach a root of A as limit. For the roots of ϕ and A satisfy the equations

$$|rI - \phi| = r^n + \rho_1 r^{n-1} + \ldots + \rho_n = 0, \tag{33}$$

$$|aI - A| = a^n + \alpha_1 a^{n-1} + \ldots + \alpha_n = 0. \tag{34}$$

The coefficients $\rho_1, \rho_2, \ldots, \rho_n$ are definite rational integral functions of the elements r_{ik} of ϕ, and the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the same functions of the elements of A. When ϕ approaches A as limit

$$\operatorname{Lim} \rho_i = \alpha_i, \quad i = 1, 2, \ldots, n.$$

Also, if the coefficient of the highest power (in this case unity) does not approach zero, the roots of a polynomial equation are continuous functions of its coefficients.* Hence as ϕ approaches A as limit the roots of (33) approach those of (34) as limits.

Since different matrices can have the same roots, the roots of ϕ can approach those of A when ϕ does not approach A as limit.

The sum $\phi_1 + \phi_2 + \ldots + \phi_m$ of m matrices is equal to a single matrix S_m. If S_m approaches a limit S when m increases indefinitely, the series

$$\phi_1 + \phi_2 + \ldots + \phi_m + \ldots$$

is said to converge and have the sum S.† In case of a multiple series, such as

$$\Sigma\phi_{i,j,k}, \quad i, j, k = 1, 2, \ldots, \infty,$$

* Weber's Algebra, Vol. I, § 44.

† Peano treated the convergence of a matrix series by means of a modulus, *Math. Annalen* (1888), Vol. XXXII, pp. 450–456. See also G. A. Bliss, *Annals of Mathematics* (2) 6, pp. 49–68.

an order of summation is assumed to be included in the definition of the series. It is thus equivalent to a simple series.

If $A, B, \ldots, P$ are commutative matrices, by Theorem III, any polynomial $f(A, B, \ldots, P)$ is expressible as a linear function of matrices depending only on $A, B, \ldots, P$, the coefficients being $f(a_i, b_i, \ldots, p_i)$ and perhaps some of the derivatives $f_{\alpha, \beta, \ldots, \rho}(a_i, b_i, \ldots, p_i)$. If these coefficients approach definite limits when the number of terms in the polynomial increases indefinitely, the matrix $f(A, B, \ldots, P)$ will then approach a definite limit.

THEOREM V. *If $A, B, \ldots, P$ are commutative matrices and $f(a, b, \ldots, p)$ an infinite series, the series $f(A, B, \ldots, P)$ will converge and be represented by* (27) *if the series $f(a_i, b_i, \ldots, p_i)$, $f_{\alpha, \beta, \ldots, \rho}(a_i, b_i, \ldots, p_i)$ in the right member of* (27) *converge.*

In this theorem it is understood that $f_{\alpha, \beta, \ldots, \rho}(a, b, \ldots, p)$ is obtained by differentiating $f(a, b, \ldots, p)$ term by term and arranging the results in the same order as the corresponding terms in $f(a, b, \ldots, p)$.

If $f(a, b, \ldots, p)$ is an analytic function of the variables $a, b, \ldots, p$ and $A, B, \ldots, P$ an equal number of commutative matrices, we define $f(A, B, \ldots, P)$ as the value (if it is definite) determined by (27). In order that the value be definite it is sufficient that $f(a, b, \ldots, p)$ be analytic near each of the points $(a_i, b_i, \ldots, p_i)$ determined by the distinct sets of associated roots. For then the derivatives in (27) are also definite. If the function is many valued, a definite branch must be chosen at each of the points $(a_i, b_i, \ldots, p_i)$. The same branch need not however be used for all.

8. *Taylor's Series.*—Let $f(z)$ be a function of the complex variable z analytic in the neighborhood of each root of a matrix A. If we choose a definite branch of $f(z)$ at each root (but not necessarily the same branch for different roots) the functions $f(A), f'(A), f''(A)$, etc. are determined by (27). Let Z be a matrix commutative with A and consider the series

$$\psi(A, Z) = f(A) + f'(A)(Z-A) + \ldots + f^m(A)\frac{(Z-A)^m}{\underline{|m}} + \ldots \tag{35}$$

Let z_i and a_i be corresponding roots of Z and A. The series (35) will converge if each root z_i lies within the circle of convergence of $f(z)$ with center at a_i. For then the series

$$f(a_i) + f'(a_i)(z_i - a_i) + \ldots + f^m(a_i)\frac{(z_i - a_i)^m}{\underline{|m}} + \ldots, \tag{36}$$

and the series obtained by substituting a_i and z_i in the partial derivatives of $\psi(x, y)$ will converge. Hence the conditions of Theorem V are satisfied.

Conversely, if (35) converges, (36) will converge, and so z_i will lie within or on the circle of convergence of $f(z)$ with center a_i. For the sum of m terms in (36) represents a root of the sum of corresponding terms in (35), and, if a matrix approaches a limit, we have shown that its roots must approach definite limits. Furthermore

$$\psi(a_i, z_i) = f(a_i) + f'(a_i)(z_i - a_i) + \ldots + f^m(a_i)\frac{(z_i - a_i)^m}{\underline{|m}} + \ldots = f(z_i).$$

Also, if

$$\psi_{\alpha,\beta}(x, y) = \frac{\partial^{\alpha+\beta}\psi(x, y)}{\partial x^\alpha \partial y^\beta}$$

it is readily seen that

$$\psi_{\alpha,0}(a_i, z_i) = f^\alpha(z_i), \quad \psi_{\alpha,\beta}(a_i, z_i) = 0, \quad \beta \neq 0.$$

Hence (27) gives the same expression for $\psi(A, Z)$ and for $f(Z)$. Therefore

$$f(Z) = f(A) + f'(A)(Z - A) + \ldots + f^m(A)\frac{(Z - A)^m}{\underline{|m}} + \ldots \tag{37}$$

We may call this the expansion of $f(Z)$ in the neighborhood of the matrix A.

THEOREM VI. *The Taylor's series* (37) *for $f(Z)$ is valid for any matrix Z commutative with A if each root of Z lies within a circle with center at the corresponding root of A, in which $f(z)$ is analytic.*

This theorem enables us to determine an analytic function of a matrix Z by means of a power series

$$c_1 + c_2(Z - A) + \ldots + c_m(Z - A)^m + \ldots,$$

and its analytic continuations just as we determine an analytic function of the complex variable z by the series

$$c_1 + c_2(z - a) + \ldots + c_m(z - a)^m + \ldots,$$

and its analytic continuations. The principal difference in the two cases is that Z and A must be commutative and that the region of convergence of the matrix series consists of m circles (one for each distinct root of A) instead of one. On a range of commutative matrices the function $f(Z)$ is one valued if each root of Z is restricted to a region of the complex plane (not necessarily the same for different roots) in which the corresponding branch of $f(z)$ is one valued.

On the Lüroth Quartic Curve.

By Frank Morley.

It has been known since 1870 * that the problem of inscribing a five-line in a planar quartic is poristic; of the ten conditions nine fall on the lines and one on the curve. Thus the quartic is one for which an invariant vanishes, and the degree of this invariant is sought. We use Aronhold's construction of a curve of class 4 from seven given points. And the starting point is the theorem of Prof. Bateman † that the seven points which have the same polar line as to a conic and a cubic give rise to a Lüroth quartic.

For completeness I indicate the proof. A conic and a cubic have the canonical forms (αx^2), (βx^3) where $(x)=0$. The polars of x are (αxy), (βxy^2). Working in a space of three dimensions the line $(y)=0$, $(\alpha xy)=0$ is to touch the quadric (βxy^2). This requires that

$$\Sigma\beta_0\beta_1 x_0 x_1(\alpha_2 x_2-\alpha_3 x_3)^2=0,$$

or

$$(\alpha/\beta)^2/(\alpha^2 x/\beta)=(1/\beta x),$$

and this is a quartic of Lüroth's type. The seven common polar lines are an Aronhold set of double lines of this quartic, and by polarity as to the conic the seven points a_i which have these polar lines are double points of a Lüroth curve of class 4.

§ 1. *The Bateman Conic.*

Take now a conic $(\alpha x)^2$ and a cubic $(\beta x)^3$. The Jacobian of these and a line (ξx)

$$(\alpha x)(\beta x)^2|\alpha\beta\xi|=0$$

gives the net of cubics on the seven points a_i. Referred to one of the points and the corresponding line let the conic be $x_0^2+2x_1x_2$ and the cubic be

$$x_0^3+x_0(\gamma x)_0^2+(\delta x)_0^3.$$

Then for $(\xi x)\equiv x_0$ the Jacobian is

$$(\alpha_1\beta_2-\alpha_2\beta_1)(\alpha x)(\beta x)^2=\beta_2(\beta x)^2x_2-\beta_1(\beta x)^2x_1,$$

so that not only terms in x_0^2 but also the term $x_0x_1x_2$ is missing.

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* Lüroth, *Math. Annalen*, Vol. I.

† American Journal of Mathematics, Vol. XXXVI.

That is, *the seven cubics with double points a_i have their nodal tangents apolar to the conic α.* I will call this conic the Bateman conic.

Given any seven points a_i, cubics on them determine by their remaining intersections a Geiser involution $a^8x^8\xi=0$. If ξ is the join of x and y, then $a^8x^9y=0$, or removing the Jacobian of the net, an a^3x^6, $a^5x^3y=0$. This is the canonic form of the net.*

It may be written as a two-one connex $a^5x^2\xi$, giving for every line ξ the Geiser pair on ξ. This Geiser pair is the neutral pair of the net of binary cubics of ξ, cut out by the net of cubic curves. The quartic, locus of lines ξ for which the Geiser pair come together, is found by making ξ touch the conic $a^5x^2\xi$, and is an $a^{10}\xi^4$.

Write the above two-one connex $a^5x^2\xi$ as $(\gamma x)^2(c\xi)$, and consider $(\gamma x)(\gamma y)\,|cxy|$ an $a^5x^2y^2$. This skew form is the polar conic of y as to its associate cubic, and when y is a_i it is the nodal tangents to the cubic with double point a_i. We have seen that in the case in question these seven line pairs are apolar to a conic. But there are only six independent conics. Thus the required condition is that for arbitrary y the associate conic be apolar to a conic. That is, the six-rowed determinant of all coefficients $\gamma_{ij}c_k$ vanishes. But being a skew determinant it is a square. Thus *a cubic function of the coefficients $\gamma_{ij}c_k$ vanishes.*

§ 2. *The Cubic Invariant of Seven Points.*

For the connex $(\gamma x)^2(c\xi)$ the possible expressions of the third degree are

$$(c\gamma)(c'\gamma')(c''\gamma'')\,|\gamma\gamma'\gamma''|,\quad (c\gamma)(c'\gamma'')(c''\gamma')\,|\gamma\gamma'\gamma''|,$$
$$(c\gamma')(c'\gamma'')(c''\gamma)\,|\gamma\gamma'\gamma''|,\quad |cc'c''|\,|\gamma\gamma'\gamma''|^2.$$

Of these the first, second, and fourth change sign on interchange of $c'\gamma'$ with $c''\gamma''$, and therefore are zero. Thus the invariant in question is

$$(c\gamma')(c'\gamma'')(c''\gamma)\,|\gamma\gamma'\gamma''|.$$

The invariant expressed in terms of the seven points a_i is of degree 15. But if six points are on a conic it will vanish.

For the form $a^5x^2y^2$ was made up in this way: on the line $\overline{xy}$ is a net of binary cubics, with a neutral pair, and xy are taken harmonic with this neutral pair. If $\overline{ya_i}$ meet the cubic curve with double point at a_i at p_i, then the

* An expression for the net of cubics on seven given points may be noted, though not of present use. Let A_1 be the Jacobian of cubics on $a_2 \ldots\ldots a_7 x$, and so on. Then the determinant of seven rows

$$|\, a^2_{i0},\ a^2_{i1},\ a^2_{i2},\ a_{i1}a_{i2},\ a_{i2}a_{i0},\ a_{i0}a_{i1},\ (yDa_i)A_i \,|$$

is the expression in question. For when $x=a_1$, $A_2 \ldots\ldots A_7$ vanish and $(yDa_1)A_1$ also vanishes. Presumably this expression for the net is canonic.

neutral pair is a_i and p_i, and if the polar of y as to these be x_i, then the seven points x_i are on the conic $a^5x^2y^2$ associate with y.

If now $a_2\ldots a_7$ are on a cônic $(\alpha x)^2$, the points $p_2\ldots p_7$ are on the line $(\alpha x)(\alpha y)$, and the form $a^5x^2y^2$ becomes $(\alpha x)(\alpha y)\,|a_1xy|$.

This with a_1 as the reference point $(1, 0, 0)$ and $(\alpha x)^2$ as $x_0^2+2x_1x_2$ becomes

$$(x_0y_0+x_1y_2+x_2y_1)(x_1y_2-x_2y_1),$$

and since there is no term in y_0^2 the invariant of the coefficients vanishes. Thus the expression of degree 15 in a_i breaks up, and removing the factors which vanish when any six of the seven points are on a conic, we are left with *a cubic expression in the* a_i.

Thus, given six of the points, the locus of the seventh, a_7, is a cubic curve. If once more the six are on a conic $(\alpha x)^2$ then the nodal tangents of the cubic $|a_ia_7x|(\alpha x)^2$ are apolar to $(\alpha x)^2$, which is therefore the Bateman conic. Then the tangents at a_7 to the cubic of the system with double point a_7 are apolar to the conic, and this defines the cubic. Thus, if the six points on the conic be given by the binary form $(\beta t)^6$, the locus of a_7 is $(\beta t)^3(\beta t')^3=0$, namely, that cubic on the six points to which the conic (as a line-curve) is apolar.

Thus a special seven-point for which the cubic invariant vanishes is six points on a conic and any point on the apolar cubic through them.

Hence, given any six points $a_1\ldots a_6$ we have a counter-six $b_1\ldots b_6$ where b_1 is the extra point in which the conic on $a_2\ldots a_6$ meets the cubic on $a_1\ldots a_6$ and apolar to this conic. The locus of a_7 passes through all twelve points. It is to be noticed that the relations of the points a_i and the points b_i are mutual.

Expressed in terms of Professor Coble's* linear invariants $\overline{a}\ldots\overline{f}$ of six points, and linear covariants $a\ldots f$, these being cubics on the points, the covariant cubic in question can be no other than $\Sigma\overline{a}^2a$. This is then an expression for the cubic invariant of seven points.

§ 3. *The Lüroth Invariant.*

If now we map the plane on a cubic surface by means of cubics on the six points $a_1\ldots a_6$, the covariant cubic curve becomes a covariant plane of the isolated double-six of lines on the surface. The construction becomes as follows: Let a_i and b_i be a pair of lines of the double-six. Sections of the surface on a_i determine points on b_i. The tangent conic sections determine two points on b_i. There is a conic on the surface through these two points and this determines a point on a_i, on the plane required.

* *Transactions*, Vol. XVI (1915), § 4.

If from any point where this plane meets the surface we draw the tangent lines we obtain a quartic curve of Lüroth's type. Now a cubic surface has thirty-six double-sixes, and therefore thirty-six such planes. The locus of points on the surface which give rise to Lüroth quartics is then thirty-six planar cubics.

But a covariant of the surface of order 2μ gives an invariant of the corresponding quartic of degree 3μ. Hence, *the Lüroth invariant is of degree* 54.

A line of the surface belongs to sixteen double sixes. Thus the thirty-six planes meet a line of the surface in $16+20$ points. Thus a Lüroth quartic can acquire a double point in two ways. In the one, the lines at the double point are apolar to the points on a double line. In the other the lines at the double point meet the curve again on a line of the curve.

This indicates the nature of the Lüroth invariant I_{54}, namely it is, to the discriminant I_{27} as modulus, the product of two invariants.

Consider a nodal cubic surface in Sylvester's form, $(\varkappa x^3)$ where $(1/\sqrt{\varkappa})=0$. It can be proved that the plane corresponding to the double-six of lines on the node is $(x\sqrt{\varkappa})$.

Hence, when $(1/\sqrt{\varkappa})$ is not 0, there is a covariant of order 16, product of the sixteen planes $(x\sqrt{\varkappa})$, meeting any line of the surface at the sixteen points on it, so that there is an invariant I_{24} which vanishes for a nodal Lüroth quartic of the first kind.

For a nodal Lüroth quartic of the second kind, the inscribed five lines are three lines on the double point, and two lines on a fixed point of the curve. In particular the tangents from the node fall into two sets of three, each set having its contacts on a line. Thus such a quartic is included in those for which three intersections of double lines lie on a line.

Looking then at a double-six from a point y of its cubic surface, the six lines from y to each pair lie on a quadric cone which breaks up into two planes when y is on one of ten planes corresponding to the separation of the six pairs into threes.

If in the case of a nodal cubic $(\varkappa x^3)$ surface these ten planes formed for the nodal double-six have an equation rational in $\varkappa_i$, then an invariant I_{15} vanishes for the nodal Lüroth quartic of the second kind, and the Lüroth invariant will be $I_{27}I_{27}^1+I_{24}I_{15}^2$ where I_{27}^1 is an invariant which is probably the discriminant also.

VINEYARD HAVEN, MASS., *July*, 1918.

*On the Order of a Restricted System of Equations.**

By F. F. Decker.

Section 1.

In an earlier paper † the writer gave a proof of the theorem—announced without proof by Salmon ‡—that the number of solutions of the system of equations arising from the vanishing of all the determinants of the m-th order that can be formed from a matrix with m rows and n columns, $m \ngtr n$, by the suppression of $n-m$ columns, the elements in the i-th row and j-th column being of degree α_i+a_j in $n-m+1$ non-homogeneous variables is K_{n-m+1} where $K_l=\sum_{i=0}^{i=l} c_i\delta_{l-i}$, c_l representing the sum of all possible products of l different a's and δ_l the sum of all possible products of l α's, repetitions being permissible.

In this paper the system arising from the vanishing of all the determinants of the r-th order that can be formed by the suppression of $n-r$ columns and $m-r$ rows, $r \ngtr m$, $r \ngtr n$, is treated. The order is shown to be

$$\begin{vmatrix} K_{n-r+1} & K_{n-r} & \dots K_{n-m+1} \\ K_{n-r+2} & K_{n-r+1} & \dots K_{n-m+2} \\ \dots & \dots & \dots \\ K_{n-2r+m+1} & K_{n-2r+m} & \dots K_{n-r+1} \end{vmatrix}. \tag{A}$$

If the degree of every element is b the order reduces to

$$\frac{\prod_{i=0}^{i=m-r} {}_{n+m-r}C_{m+1-i}}{\prod_{i=0}^{i=m-r} {}_{n+m-r}C_i} \cdot b^{(n-r+1)(m-r+1)}, \text{ which may be written}$$

$$\frac{\prod_{i=0}^{i=m-r} {}_{n+i}C_{n-r+1}}{\prod_{i=0}^{i=m-r} {}_{n-r+1+i}C_{n-r+1}} \cdot b^{(n-r+1)(m-r+1)}, \tag{B}$$

a result established by Segre. §

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* Presented before the American Mathematical Society, April 27, 1918.

† "On the Order of a Restricted System of Equations," American Journal of Mathematics, Vol. XXXVII, No. 2, April, 1915.

‡ Salmon, "Modern Algebra," Fourth Edition, pp. 283-313.

§ Segre, "Gli ordini delle varietà che annullano dei diversi gradi estratti da una data matrice," *Rendic. R. Accad. Dei Lincei*, Series 5, Vol. IX, session of October 21, 1900.

In § 2 the notation is defined and some preliminary relations are given. Theorem VI simplifies the derivation of the order for $m > n$ after it has been derived for $m \ngtr n$ (Theorem XII). In § 3 is considered the case of the vanishing of the determinants of two matrices having some common rows. The order of that part of the system for which the determinants of the common rows vanish is found (Theorem IX). In § 4 is established by mathematical induction, with the aid to Theorem V, the relation (A) of § 1 (Theorem XIII), and the relation (B) is established in § 5 (Theorem XIV).

Section 2.

$\left\| \begin{matrix} u_{11} \dots u_{1n} \\ \dots\dots\dots \\ u_{m1} \dots u_{mn} \end{matrix} \right\|^{(r)}$ or $\| U_{mn} \|^{(r)}$ or $\begin{pmatrix} 1 \dots n \\ 1 \dots m \end{pmatrix}^{(r)}$ denotes the aggregate of all determinants of the r-th order that can be formed from the matrix $\left\| \begin{matrix} u_{11} \dots u_{1n} \\ \dots\dots\dots \\ u_{m1} \dots u_{mn} \end{matrix} \right\|$ by suppressing $m-r$ rows and $n-r$ columns. That all these determinants vanish will be indicated by

$$\left\| \begin{matrix} u_{11} \dots u_{1n} \\ \dots\dots\dots \\ u_{m1} \dots u_{mn} \end{matrix} \right\|^{(r)} = 0 \text{ or } \| U_{mn} \|^{(r)} = 0 \text{ or } \begin{pmatrix} 1 \dots n \\ 1 \dots m \end{pmatrix}^{(r)} = 0.$$

$M(\bar{h}, \bar{k})$ denotes the matrix formed from $\begin{pmatrix} 1 \dots n \\ 1 \dots m \end{pmatrix}$ by suppressing the first h rows and the last k rows, that is $\begin{pmatrix} 1 \dots n \\ h+1 \dots m-k \end{pmatrix}$.

The order of the system of equations $A=0$ will be indicated by $\bar{A}$. When the (r) is lacking from the symbol it will be understood to be the same as the smaller of the two numbers m, n. If r is one more than the smaller of the numbers m, n, the order symbol is to be understood to represent the number 1, and if r exceeds each of the numbers m, n by more than one, the order symbol is to be understood to represent zero.

ϕ is an operator such the $\phi \bar{M}(\bar{h}_1, \bar{k}_1) \bar{M}(\bar{h}_2, \bar{k}_2) \dots \bar{M}(\bar{h}_s, \bar{k}_s)$ produces the algebraic sum of all the products that can be formed from

$$\bar{M}(\bar{h}_1, \bar{k}_1) \bar{M}(\bar{h}_2, \bar{k}_2) \dots \bar{M}(\bar{h}_s, \bar{k}_s)$$

by interchanging k_1 with each of the other k's in turn, the sign of each product being given by the formula $(-1)^{t_i+1}$ where t_i is the number of inversions of

the natural order of the k's in the different permutations, the k's occurring in the natural order. $K_l \equiv \sum_{i=0}^{i=l} c_i \delta_{l-i}$ and $J_l \equiv \sum_{i=0}^{i=l} d_i \gamma_{l-i}$ where $c_l \equiv \sum_1^s \prod_{i=1}^{i=l} a_i$, $\gamma_l \equiv \sum_1^s \prod_{i=1}^{i=l} \alpha_i$, and d and δ differ from c and γ, respectively, only in permitting repetitions of the same letter in forming homogeneous products. $K_0 = J_0 = c_0 = d_0 = \gamma_0 = \delta_0 = 1$. When l is negative the values of the symbols are zero. $K'_s = K_{n-m+s}$.

u_{rs} denotes a function of not less than $(n-r+1)(m-r+1)$ non-homogeneous variables, and, whenever the order is calculated in terms of K's or J's, u_{rs} is considered to be of degree $a_r + a_s$.

Use will be made of the following relations (Theorems I–III), proofs of which may be found in a previous paper on the subject by the writer already referred to.

THEOREM I. $\overline{\begin{pmatrix} 1 \ldots n \\ 1 \ldots m \end{pmatrix}} = K_{n-m+1}$.

THEOREM II. $$\overline{\begin{pmatrix} 1 \ldots n \\ 1 \ldots m \end{pmatrix}} = \sum_1^m (-1)^{i+1} \overline{\begin{pmatrix} i+1 \ldots n \\ 1 \ldots m \end{pmatrix}}^{(m)} \overline{\begin{pmatrix} i \ldots m \\ 1 \ldots m \end{pmatrix}}^{(m-i+1)}$$
$$= \sum_1^m (-1)^{i+1} \overline{\begin{pmatrix} 1 \ldots n-i \\ 1 \ldots m \end{pmatrix}}^{(m)} \overline{\begin{pmatrix} n-m+1 \ldots n-i+1 \\ 1 \ldots m \end{pmatrix}}^{(m-i+1)}.$$

THEOREM III. $\sum_{i=0}^{i=l} (-1)^i K_{l-i} J_i = 0$.

THEOREM IV. $\underset{a=r+1\ldots m}{K_n} = K_n - K_{n-1} \sum_1^r \alpha_1 + K_{n-2} \sum_1^r \alpha_1\alpha_2 - \ldots + (-1)^r K_{n-r} \sum_1^r \alpha_1\alpha_2 \ldots \alpha_r$, *where the a's run from* 1 *to* n *in all the K's, and the α's run from* 1 *to* m, *except in the case noted.*

To establish this relation we divide the terms of K_n into those that contain α_1 and those that do not. From the definition of K, $K_n = \alpha_1 K_{n-1} + \underset{a=2\ldots m}{K_n}$, whence $\underset{a=2\ldots m}{K_n} = K_n - \alpha_1 K_{n-1}$. Then we assume that

$$\underset{r\ldots m}{K_n} = K_n - K_{n-1} \sum_1^{r-1} \alpha_1 + \ldots + (-1)^{l-1} K_{n-l+1} \sum_1^{r-1} \alpha_1\alpha_2 \ldots \alpha_{l-1}$$
$$+ (-1)^l K_{n-l} \sum_1^{r-1} \alpha_1\alpha_2 \ldots \alpha_l + \ldots.$$

It follows that

$$\underset{r+1\ldots m}{K_n} = \underset{r\ldots m}{K_n} - \alpha_r \underset{r\ldots m}{K_{n-1}} = K_n - (\alpha_r + \sum_1^{r-1} \alpha_1) K_{n-1}$$
$$+ \ldots + (-1)^s (\alpha_r \sum_1^{r-1} \alpha_1\alpha_2 \ldots \alpha_{s-1} + \sum_1^{r-1} \alpha_1\alpha_2 \ldots \alpha_s) K_{n-s} + \ldots$$
$$= K_n - K_{n-1} \sum_1^r \alpha_1 + \ldots + (-1)^s K_{n-s} \sum_1^r \alpha_1\alpha_2 \ldots \alpha_s$$
$$+ \ldots + (-1)^r K_{n-r} \sum_1^r \alpha_1\alpha_2 \ldots \alpha_r.$$

THEOREM V. $(1-\phi)\begin{pmatrix}\overline{1\ldots n}\\ s+1\ldots m\end{pmatrix}\Delta_s=\Delta_{s+1},\ m \not> n,$

$$\text{where } \Delta_l=\begin{vmatrix}\begin{pmatrix}\overline{1\ldots n}\\ 1\ldots m-l\end{pmatrix} & \begin{pmatrix}\overline{1\ldots n}\\ 1\ldots m-l+1\end{pmatrix} & \ldots & \begin{pmatrix}\overline{1\ldots n}\\ 1\ldots m-1\end{pmatrix}\\ \begin{pmatrix}\overline{1\ldots n}\\ 2\ldots m-l\end{pmatrix} & \begin{pmatrix}\overline{1\ldots n}\\ 2\ldots m-l+1\end{pmatrix} & \ldots & \begin{pmatrix}\overline{1\ldots n}\\ 2\ldots m-1\end{pmatrix}\\ \ldots & \ldots & \ldots & \ldots\\ \begin{pmatrix}\overline{1\ldots n}\\ l\ldots m-l\end{pmatrix} & \begin{pmatrix}\overline{1\ldots n}\\ l\ldots m-l+1\end{pmatrix} & \ldots & \begin{pmatrix}\overline{1\ldots n}\\ l\ldots m-1\end{pmatrix}\end{vmatrix}$$

$$=\begin{vmatrix}\overline{M}(\overline{0},\overline{l}) & \overline{M}(\overline{0},\overline{l-1}) & \ldots\overline{M}(\overline{0},\overline{1})\\ \overline{M}(\overline{1},\overline{l}) & \overline{M}(\overline{1},\overline{l-1}) & \ldots\overline{M}(\overline{1},\overline{1})\\ \ldots & \ldots & \ldots\\ \overline{M}(\overline{l-1},\overline{l}) & \overline{M}(\overline{l-1},\overline{l-1}) & \ldots\overline{M}(\overline{l-1},\overline{1})\end{vmatrix}.$$

Proof: $\Delta_s=\Sigma(-1)^{t_i+1}\overline{M}(0,k_1)\overline{M}(\overline{1},\overline{k}_2)\ldots\overline{M}(\overline{s-1},\overline{k}_s)$, where k_1, $k_2,\ldots,k_s$ is a permutation of the numbers $1, 2, \ldots, s$, and t_i the number of inversions of the natural order of the k's in the different permutations. When ϕ operates on $\overline{M}(\overline{s},\overline{0})\Delta_s$ and the series of first indices is restored to the natural order, all the permutations of the second indices except $k_1, k_2, \ldots, k_s, 0$ will be obtained and the number of inversions of the series of second indices will be changed by one. Therefore

$$(1-\phi)\overline{M}(\overline{s},\overline{0})\Delta_s=\Sigma(-1)^{t_i+1}\overline{M}(\overline{0},\overline{k}_1)\overline{M}(\overline{1},\overline{k}_2)\ldots\overline{M}(\overline{s},\overline{k}_{s+1})=\Delta_{s+1}.$$

THEOREM VI. $\Delta=\Delta'$,*

$$\text{where } \Delta=\begin{vmatrix}K_s & K_{s-1} & \ldots K_1\ 1\ \ 0\ 0\ldots 0 & 0\\ K_{s+1} & K_s & \ldots K_2\ K_1\ 1\ 0\ldots 0 & 0\\ \ldots & \ldots & \ldots & \ldots\\ K_{t-1} & K_{t-2} & \ldots\ldots K_1 & 1\\ K_t & K_{t-1} & \ldots\ldots K_2 & K_1\\ \ldots & \ldots & \ldots & \ldots\\ K_{s+t-1} & K_{s+t-2} & \ldots\ldots K_{s+1} & K_s\end{vmatrix} \text{ and } \Delta'=\begin{vmatrix}J_t & J_{t-1} & \ldots J_{t-s+1}\\ J_{t+1} & J_t & \ldots J_{t-s+2}\\ \ldots & \ldots & \ldots\\ J_{t+s-1} & J_{t+s-2} & \ldots J_t\end{vmatrix}$$

and $s \not> t$.

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* For the case $m=0$, Theorem VI yields a relation between the total symmetric functions and the elementary products of the a's. If also $s=1$, there results a formula for the total symmetric function H_t in terms of the elementary products, a formula which has already been proved by Roe in the *Transactions of the American Mathematical Society*, Vol. V, No. 2, p. 202, April, 1904.

$$\Delta^{\mathrm{II}} = \left|\begin{array}{llllllll}
0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & K_1 & 1 & \ldots & 0 \\
\cdot & \cdot & \ldots & \cdot & \cdot & \cdot & \ldots & \cdot \\
0 & 0 & \ldots & 0 & \cdot & \cdot & \ldots & K \\
(-1)^{t-1}J_t & (-1)^{t-2}J_{t-1} & \ldots & (-1)^{t-s}J_{t-s+1} & \cdot & \cdot & \ldots & K \\
(-1)^{t-1}(K_1J_t-J_{t+1}) & (-1)^{t-2}(K_1J_{t-1}-J_t) & \ldots & (-1)^{t-s}(K_1J_{t-s+1}-J_{t-s+2}) & \cdot & \cdot & \ldots & K_s \\
\ldots & \ldots & \ldots & \ldots & \cdot & \cdot & \ldots & \ldots \\
\sum_{t}^{s+t-1}(-1)^{i-1}K_{s+t-1-i}J_i & \sum_{t-1}^{s+t-2}(-1)^{i-1}K_{s+t-2-i}J_i & \ldots & \sum_{t-s+1}^{t}(-1)^{i-1}K_{t-i}J_i & K_{t-1} & K_{t-2} & \ldots & K
\end{array}\right|$$

$$\Delta^{\mathrm{III}} = \left|\begin{array}{llll}
(-1)^{t-1}J_t & (-1)^{t-2}J_{t-1} & \ldots & (-1)^{t-s}J_{t-s+1} \\
(-1)^{t-1}(K_1J_t-J_{t+1}) & (-1)^{t-2}(K_1J_{t-1}-J_t) & \ldots & (-1)^{t-s}(K_1J_{t-s+1}-J_{t-s+2}) \\
\ldots & \ldots & \ldots & \ldots \\
\sum_{t}^{s+t-1}(-1)^{i-1}K_{s+t-1-i}J_i & \sum_{t-1}^{s+t-2}(-1)^{i-1}K_{s+t-2-i}J_i & \ldots & \sum_{t-s+1}^{t}(-1)^{i-1}K_{t-i}J_i
\end{array}\right|,$$

$$\Delta^{\mathrm{IV}} = \left|\begin{array}{llll}
+J_t & +J_{t-1} & \ldots & +J_{t-s+1} \\
-J_{t+1} & -J_t & \ldots & -J_{t-s+2} \\
+J_{t+2} & +J_{t+1} & \ldots & +J_{t-s+8} \\
\ldots & \ldots & \ldots & \ldots \\
(-1)^{s-1}J_{t+s-1} & (-1)^{s-1}J_{t+s-2} & \ldots & (-1)^{s-1}J_t
\end{array}\right| \text{ and } D = \left|\begin{array}{llll}
1 & 0 & \ldots & 0 \\
K_1 & 1 & \ldots & 0 \\
\cdot\cdot & \cdot & \ldots & \cdot \\
K_{s-1} & K_{s-2} & \ldots & 1
\end{array}\right| = 1.$$

also let $l=(t-s)s$,

$$p=l+\frac{s}{2}(2t-s-1),\quad q=\begin{cases} p+\dfrac{s}{2}, & \text{when } s \text{ is even,} \\ p+\dfrac{s-1}{2}, & \text{when } s \text{ is odd,}\end{cases}$$

q therefore always being even.

If for the K's of the first column of Δ the values given by Theorem III are substituted and the result expressed as the algebraic sum of $s+t-1$ determinants by using the corresponding terms of the elements of the first column for the first columns, then the process repeated for the second, third, ..., s-th column in turn, the sum of the determinants that do not vanish on account of columns differing by a common factor is identical with Δ^{II}. If Δ^{II} is expanded by Laplace's development in terms of its first $t-s$ rows, the result is $(-1)^l\cdot\Delta^{\mathrm{III}}$. After the factor $(-1)^{t-1}$ is removed from the first column, $(-1)^{t-2}$ from the second, and so on, the resulting determinant is, according to the product theorem for determinants, $D\cdot\Delta^{\mathrm{IV}}$. Thus

$$\Delta=(-1)^p\cdot D\cdot\Delta^{\mathrm{IV}}=(-1)^q\cdot\Delta'=\Delta'.$$

Section 3.

$\overline{\begin{pmatrix}1\ldots n\\1\ldots m\end{pmatrix}}^{(m-1)}$ will now be considered. $\begin{pmatrix}1\ldots n\\1\ldots m\end{pmatrix}^{(m-1)}\neq 0$ unless $\begin{pmatrix}1\ldots n\\1\ldots m-1\end{pmatrix}=0$ and $\begin{pmatrix}1\ldots n\\2\ldots m\end{pmatrix}=0$. It vanishes with them except when $\begin{pmatrix}1\ldots n\\2\ldots m-1\end{pmatrix}=0$.* Let $\overline{\begin{pmatrix}1\ldots n\\1\ldots m\end{pmatrix}}^{(m-1)}=\overline{\begin{pmatrix}1\ldots n\\1\ldots m-1\end{pmatrix}}\cdot\overline{\begin{pmatrix}1\ldots n\\2\ldots m\end{pmatrix}}-\overline{X}$. The value of $\overline{X}$ will now be found, or rather a quantity satisfying a more general condition will be found (Theorem IX), and from it the value of $\overline{X}$ will follow as a special case. First, however, some preliminary theorems (VII and VIII) will be proved.

THEOREM VIIa. $\overline{\begin{pmatrix}1\ldots 6\\1\ldots 5\end{pmatrix}}=\overline{\begin{pmatrix}123\\12\end{pmatrix}}+\overline{\begin{pmatrix}12\\12\end{pmatrix}}\cdot\overline{\begin{pmatrix}456\\345\end{pmatrix}}+\overline{\begin{pmatrix}3456\\345\end{pmatrix}}$.

This relation will be proved by a system of sections. The section of $\begin{pmatrix}1\ldots 6\\1\ldots 5\end{pmatrix}=0$ by the spreads $\begin{pmatrix}1\\2\ldots 5\end{pmatrix}=0$ degenerates into two parts of the same dimension for one of which, say A, $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}2\ldots 6\\1\ldots 5\end{pmatrix}$ also vanish, and for the other $B=\begin{pmatrix}2\ldots 6\\2\ldots 5\end{pmatrix}=0$. Thus $\overline{\begin{pmatrix}1\ldots 6\\1\ldots 5\end{pmatrix}}=\overline{A}+\overline{B}$. Again, taking the section of $A=0$ by $\begin{pmatrix}23\\345\end{pmatrix}^{(1)}=0$, the section of $B=0$ by $\begin{pmatrix}2\\345\end{pmatrix}=0$, and then taking the section of one of the parts into which $B=0$ degenerates by $\begin{pmatrix}3\\345\end{pmatrix}=0$, it is found that

$$\overline{\begin{pmatrix}1\ldots 6\\1\ldots 5\end{pmatrix}}=\overline{\begin{pmatrix}1\\1\end{pmatrix}}\left[\overline{\begin{pmatrix}23\\12\end{pmatrix}}+\overline{\begin{pmatrix}456\\345\end{pmatrix}}\right]+\overline{\begin{pmatrix}2\\2\end{pmatrix}}\left[\overline{\begin{pmatrix}3\\2\end{pmatrix}}+\overline{\begin{pmatrix}456\\345\end{pmatrix}}\right]+\overline{\begin{pmatrix}3\ldots 6\\345\end{pmatrix}}$$

$$=\overline{C}+\overline{D}+\overline{E};\text{ where }\overline{C}=\overline{\begin{pmatrix}1\\1\end{pmatrix}}\overline{\begin{pmatrix}23\\12\end{pmatrix}}+\overline{\begin{pmatrix}23\\2\end{pmatrix}},$$

$$\overline{D}=\left[\overline{\begin{pmatrix}1\\1\end{pmatrix}}+\overline{\begin{pmatrix}2\\2\end{pmatrix}}\right]\overline{\begin{pmatrix}456\\345\end{pmatrix}}\text{ and }\overline{E}=\overline{\begin{pmatrix}3\ldots 6\\345\end{pmatrix}}.$$

Now $\overline{C}$ is seen to give the first term required and $\overline{D}$ the second, by taking sections by $\begin{pmatrix}1\\2\end{pmatrix}=0$. The theorem follows.

THEOREM VII. $\overline{\begin{pmatrix}1\ldots n\\1\ldots m\end{pmatrix}}=\sum_{i=n-m+k}^{i=k-1}\overline{\begin{pmatrix}1\ldots i\\1\ldots k\end{pmatrix}}^{(k)}\overline{\begin{pmatrix}i+2\ldots n\\k+1\ldots m\end{pmatrix}}^{(m-k)}$.

* See author's paper already referred to.

Reducing $\overline{\binom{1\ldots k+1}{1\ldots s}}^{(s)}$ by Theorem II,

$$\begin{aligned}\overline{\binom{1\ldots n}{1\ldots s}}^{(s)} = {} & \overline{\binom{k-s+3\ldots n}{1\ldots s}}^{(s)}\Big[\overline{\binom{1\ldots k}{1\ldots s}}^{(s)}\overline{\binom{k-s+2\ldots k+1}{1\ldots s}}^{(s)} \\ & -\overline{\binom{1\ldots k-1}{1\ldots s}}^{(s)}\overline{\binom{k-s+2\ldots k}{1\ldots s}}^{(s-1)}+\ldots\Big] \\ & +\overline{\binom{1\ldots k}{1\ldots s}}^{(s)}\Big[\overline{\binom{k-s+2\ldots n}{1\ldots s}}^{(s)} \\ & -\overline{\binom{k-s+2\ldots k+1}{1\ldots s}}^{(s)}\overline{\binom{k-s+3\ldots n}{1\ldots s}}^{(s)}\Big]+\ldots \\ = {} & \overline{\binom{1\ldots k}{1\ldots s}}^{(s)}\overline{\binom{k-s+2\ldots n}{1\ldots s}}^{(s)}+\overline{\binom{1\ldots k-1}{1\ldots s}}^{(s)}\big[\;\big]+\ldots.\end{aligned}$$

Hence $\overline{\binom{1\ldots k}{1\ldots s}}^{(s)}\overline{\binom{k-s+2\ldots n}{1\ldots s}}^{(s)}=\overline{\binom{1\ldots n}{1\ldots s}}^{(s)}+F$, where none of the matrices in F contain the n-th column. Therefore $\bar{Y}_{j,k}=\bar{Y}_{k-s+2,k}=\overline{\binom{1\ldots n}{1\ldots s}}^{(s)}$.

THEOREM VIIId. *If* $j=k-s+1$,

$$\bar{Y}_{j,k}=\bar{Y}_{k-s+1,k}=\overline{\binom{k-s+1\ldots n}{1\ldots s}}^{(s)}\overline{\binom{1\ldots n}{1\ldots s}}^{(s)}.$$

Proof:

$$\begin{aligned}\overline{\binom{1\ldots k}{1\ldots s}}^{(s)}\overline{\binom{k-s+1\ldots n}{1\ldots s}}^{(s)} = \overline{\binom{1\ldots k}{1\ldots s}}\Big[& \overline{\binom{k-s+2\ldots n}{1\ldots s}}^{(s)}\overline{\binom{k-s+1\ldots k}{1\ldots s}}^{(s)} \\ & -\overline{\binom{k-s+3\ldots n}{1\ldots s}}^{(s)}\overline{\binom{k-s+2\ldots k}{1\ldots s}}^{(s-1)} \\ & +\overline{\binom{k-s+4\ldots n}{1\ldots s}}^{(s)}\overline{\binom{k-s+3\ldots k}{1\ldots s}}^{(s-2)}-\ldots\Big].\end{aligned}$$

Therefore

$$\begin{aligned}\bar{Y}_{k-s+1,k}= {} & \overline{\binom{k-s+1\ldots k}{1\ldots s}}^{(s)}\bar{Y}_{k-s+2,k} \\ & -\overline{\binom{k-s+2\ldots k}{1\ldots s}}^{(s-1)}\bar{Y}_{k-s+3,k}+\overline{\binom{k-s+3\ldots k}{1\ldots s}}^{(s-2)}\bar{Y}_{k-s+4,k}-\ldots.\end{aligned}$$

But $\bar{Y}_{k-s+2,k}=\overline{\binom{1\ldots n}{1\ldots s}}^{(s)}$ by Theorem VIIIc and $\bar{Y}_{k-s+3,k}=\bar{Y}_{k-s+4,k}=\ldots=0$ by Theorem VIIIa. Therefore

$$\bar{Y}_{j,k}=\bar{Y}_{k-s+1,k}=\overline{\binom{k-s+1\ldots k}{1\ldots s}}^{(s)}\overline{\binom{1\ldots n}{1\ldots s}}^{(s)}.$$

THEOREM VIII. *The order* $\overline{Y}_{j,k}$ *of the part of the system* $\binom{1\ldots k}{1\ldots s}=0$, $\binom{j\ldots n}{1\ldots s}^{(s)}=0$ *for which* $\binom{1\ldots n}{1\ldots s}^{(s)}=0$ *is*

$$\overline{\binom{j\ldots k}{1\ldots s}}^{(s)}\overline{\binom{1\ldots n}{1\ldots s}}^{(s)},\quad s \not> j \not> k \not> n.$$

Proof: Suppose it was already shown that

$$\overline{Y}_{j+i,k}=\overline{\binom{j+i\ldots k}{1\ldots s}}^{(s)}\overline{\binom{1\ldots n}{1\ldots s}}^{(s)},\quad i=1\ldots k-j.$$

Since $\overline{\binom{1\ldots k}{1\ldots s}}^{(s)}\overline{\binom{j\ldots n}{1\ldots s}}^{(s)}$

$$=\overline{\binom{1\ldots k}{1\ldots s}}^{(s)}\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\binom{j+i\ldots n}{1\ldots s}}^{(s)}\overline{\binom{j-1+i\ldots j+s-1}{1\ldots s}}^{(s-i+1)},$$

by Theorem II,

$$=\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\binom{j-1+i\ldots j+s-1}{1\ldots s}}^{(s-i+1)}\left[\overline{\binom{1\ldots k}{1\ldots s}}^{(s)}\overline{\binom{j+i\ldots n}{1\ldots s}}^{(s)}\right].$$

Therefore

$$\begin{aligned}
\overline{Y}_{j,k}&=\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\binom{j-1+i\ldots j+s-1}{1\ldots s}}^{(s-i+1)}\overline{Y}_{j+i,k}\\
&=\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\binom{j-1+i\ldots j+s-1}{1\ldots s}}^{(s-i+1)}\overline{\binom{j+i\ldots k}{1\ldots s}}^{(s)}\overline{\binom{1\ldots n}{1\ldots s}}^{(s)}\\
&=\overline{\binom{1\ldots n}{1\ldots s}}^{(s)}\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\binom{j+i\ldots k}{1\ldots s}}^{(s)}\overline{\binom{j-1+i\ldots j+s-1}{1\ldots s}}^{(s-i+1)}\\
&=\overline{\binom{j\ldots k}{1\ldots s}}^{(s)}\overline{\binom{1\ldots n}{1\ldots s}}^{(s)},\text{ by Theorem II.}
\end{aligned}$$

It is to be noticed that Theorems VIIIa, b, c are special cases of Theorem VIII, since by definition of the order symbol $\overline{\binom{j\ldots k}{1\ldots s}}^{(s)}$ has the value 1 when $j=k-s+2$, and the value 0 when $j>k-s+2$.

THEOREM IX. *The order* $\overline{X}_{h,l}$ *of the part of the system* $\binom{1\ldots n}{1\ldots l}^{(l)}=0$, $\binom{1\ldots n}{h\ldots m}^{(m-h+1)}=0$, *for which* $\binom{1\ldots n}{h\ldots l}^{(l-h+1)}=0$ *is*

$$\overline{\binom{1\ldots n}{h\ldots l}}^{(l-h+1)}\overline{\binom{1\ldots n}{1\ldots m}}^{(m)},\quad h \not> l \not> m \not> n.$$

Proof: $\overline{\binom{1\ldots n}{1\ldots l}}^{(l)} = \sum_{i=n-l+h-1}^{i=h-2} \overline{\binom{1\ldots i}{1\ldots h-1}}^{(h-1)} \overline{\binom{i+2\ldots n}{h\ldots l}}^{(l-h+1)}$

and $\overline{\binom{1\ldots n}{h\ldots m}}^{(m-h+1)} = \sum_{j=n-m+l}^{j=l-h} \overline{\binom{1\ldots j}{h\ldots l}}^{(l-h+1)} \overline{\binom{j+2\ldots n}{l+1\ldots m}}^{(m-l)}$,

according to Theorem VII. Therefore

$$\overline{\binom{1\ldots n}{1\ldots l}}^{(l)} \overline{\binom{1\ldots n}{h\ldots m}}^{(m-h+1)}$$

$$= \sum_{i=n-l+h-1}^{i=h-2} \overline{\binom{1\ldots i}{1\ldots h-1}}^{(h-1)} \overline{\binom{i+2\ldots n}{h\ldots l}}^{(l-h+1)} \sum_{j=n-m+l}^{j=l-h} \overline{\binom{1\ldots j}{h\ldots l}}^{(l-h+1)} \overline{\binom{j+2\ldots n}{l+1\ldots m}}^{(m-l)}$$

$$= \sum_{i=j-l+h-1}^{i=h-2} \overline{\binom{1\ldots i}{1\ldots h-1}}^{(h-1)} \cdot \sum_{j=n-m+l}^{j=l-h} \overline{\binom{1\ldots j}{h\ldots l}}^{(l-h+1)} \overline{\binom{i+2\ldots n}{h\ldots l}}^{(l-h+1)} \overline{\binom{j+2\ldots n}{l+1\ldots m}}^{(m-l)}.$$

Therefore

$$\overline{X}_{h,l} = \sum_{i=j-l+h-1}^{i=h-2} \overline{\binom{1\ldots i}{1\ldots h-1}}^{(h-1)} \cdot \sum_{j=n-m+l}^{j=l-h} \overline{Y}_{i+2,j} \overline{\binom{j+2\ldots n}{l+1\ldots m}}^{(m-l)}$$

$$= \sum_{i=j-l+h-1}^{i=h-2} \overline{\binom{1\ldots i}{1\ldots h-1}}^{(h-1)} \cdot \sum_{j=n-m+l}^{j=l-h} \overline{\binom{i+2\ldots j}{h\ldots l}}^{(l-h+1)} \overline{\binom{1\ldots n}{h\ldots l}}^{(l-h+1)} \overline{\binom{j+2\ldots n}{l+1\ldots m}}^{(m-l)},$$

by Theorem VIII,

$$= \overline{\binom{1\ldots n}{h\ldots l}}^{(l-h+1)} \sum_{i=j-l+h-1}^{i=h-2} \overline{\binom{1\ldots i}{1\ldots h-1}}^{(h-1)} \cdot \sum_{j=n-m+l}^{j=l-h} \overline{\binom{i+2\ldots j}{h\ldots l}}^{(l-h+1)} \overline{\binom{j+2\ldots n}{l+1\ldots m}}^{(m-l)}$$

$$= \overline{\binom{1\ldots n}{h\ldots l}}^{(l-h+1)} \sum_{i=j-l+h-1}^{i=h-2} \overline{\binom{1\ldots i}{1\ldots h-1}}^{(h-1)} \overline{\binom{i+2\ldots n}{h\ldots m}}^{(m-h+1)}, \text{ by Theorem VII,}$$

$$= \overline{\binom{1\ldots n}{h\ldots l}}^{(l-h+1)} \overline{\binom{1\ldots n}{1\ldots m}}^{(m)}, \text{ by Theorem VII.}$$

COROLLARY: $\overline{X}_{h,h} = \phi \overline{\binom{1\ldots n}{1\ldots l}}\,\overline{\binom{1\ldots n}{h\ldots m}}$, *according to the definition of* ϕ.

Section 4.

THEOREM X. *If* $m \ngtr n$, $\overline{\binom{1\ldots n}{1\ldots m}}^{(m-1)} = \begin{vmatrix} \overline{\binom{1\ldots n}{1\ldots m-1}} & \overline{\binom{1\ldots n}{1\ldots m}} \\ \overline{\binom{1\ldots n}{2\ldots m-1}} & \overline{\binom{1\ldots n}{2\ldots m}} \end{vmatrix}$.

Proof: As previously pointed out $\overline{\binom{1\ldots n}{1\ldots m}}^{(m-1)} = \overline{\binom{1\ldots n}{1\ldots m-1}}\,\overline{\binom{1\ldots n}{2\ldots m}}$ $-\overline{X}_{2,m-1} = \overline{\binom{1\ldots n}{1\ldots m-1}}\,\overline{\binom{1\ldots n}{2\ldots m}} - \overline{\binom{1\ldots n}{1\ldots m}}\,\overline{\binom{1\ldots n}{2\ldots m-1}}$, by Theorem IX.

The theorem follows.

COROLLARY: $\left(\begin{matrix}\overline{1\ldots n}\\ 1\ldots m\end{matrix}\right)^{(m-1)}=\begin{vmatrix}K'_2 & K'_1\\ K'_3 & K'_2\end{vmatrix}$, *where* $K'_i=K_{n-m+i}$, $m \not> n$.

Proof: $$\left(\begin{matrix}\overline{1\ldots n}\\ 1\ldots m\end{matrix}\right)^{(m-1)}=\begin{vmatrix}\underset{a=1\ldots m-1}{K'_2} & \underset{a=1\ldots m}{K'_1}\\ \underset{a=2\ldots m-1}{K'_3} & \underset{a=2\ldots m}{K'_2}\end{vmatrix}, \text{ by theorems X and I,}$$

$$=\begin{vmatrix}\underset{a=1\ldots m}{K'_2-a_mK'_1} & \underset{a=1\ldots m}{K'_1}\\ \underset{a=2\ldots m}{K'_3-a_mK'_2} & \underset{a=2\ldots m}{K'_2}\end{vmatrix}, \text{ by Theorem IV,}$$

$$=\begin{vmatrix}\underset{a=1\ldots m}{K'_2} & \underset{a=1\ldots m}{K'_1}\\ \underset{a=2\ldots m}{K'_3} & \underset{a=2\ldots m}{K'_2}\end{vmatrix}.$$

By introducing a_1 into the second row in a similar way, the corollary follows.

THEOREM XI. *If* $m \not> n \left(\begin{matrix}\overline{1\ldots n}\\ 1\ldots m\end{matrix}\right)^{(m-2)}$

$$=\begin{vmatrix}\left(\begin{matrix}\overline{1\ldots n}\\ 1\ldots m-2\end{matrix}\right) & \left(\begin{matrix}\overline{1\ldots n}\\ 1\ldots m-1\end{matrix}\right) & \left(\begin{matrix}\overline{1\ldots n}\\ 1\ldots m\end{matrix}\right)\\ \left(\begin{matrix}\overline{1\ldots n}\\ 2\ldots m-2\end{matrix}\right) & \left(\begin{matrix}\overline{1\ldots n}\\ 2\ldots m-1\end{matrix}\right) & \left(\begin{matrix}\overline{1\ldots n}\\ 2\ldots m\end{matrix}\right)\\ \left(\begin{matrix}\overline{1\ldots n}\\ 3\ldots m-2\end{matrix}\right) & \left(\begin{matrix}\overline{1\ldots n}\\ 3\ldots m-1\end{matrix}\right) & \left(\begin{matrix}\overline{1\ldots n}\\ 3\ldots m\end{matrix}\right)\end{vmatrix}= \textit{say } \Delta_3.$$

Proof: $\left(\begin{matrix}\overline{1\ldots n}\\ 1\ldots m\end{matrix}\right)^{(m-2)}\neq 0$ unless $\left(\begin{matrix}1\ldots n\\ 1\ldots m-1\end{matrix}\right)^{(m-2)}=0$ and $\left(\begin{matrix}1\ldots n\\ 3\ldots m\end{matrix}\right)=0$.

It vanishes with them except when $\left(\begin{matrix}1\ldots n\\ 3\ldots m-1\end{matrix}\right)=0.$*

Hence $$\left(\begin{matrix}\overline{1\ldots n}\\ 1\ldots m\end{matrix}\right)^{(m-2)}=\left(\begin{matrix}\overline{1\ldots n}\\ 1\ldots m-1\end{matrix}\right)^{(m-2)}\left(\begin{matrix}\overline{1\ldots n}\\ 3\ldots m\end{matrix}\right)-\bar{X}_{3,\,m-1}$$

$$=(1-\phi)\left(\begin{matrix}\overline{1\ldots n}\\ 3\ldots m\end{matrix}\right)\left(\begin{matrix}\overline{1\ldots n}\\ 1\ldots m-1\end{matrix}\right)^{(m-2)},$$

by Theorem IX, corollary,

$$=(1-\phi)\left(\begin{matrix}\overline{1\ldots n}\\ 3\ldots m\end{matrix}\right)\begin{vmatrix}\left(\begin{matrix}\overline{1\ldots n}\\ 1\ldots m-1\end{matrix}\right) & \left(\begin{matrix}\overline{1\ldots n}\\ 1\ldots m\end{matrix}\right)\\ \left(\begin{matrix}\overline{1\ldots n}\\ 2\ldots m-1\end{matrix}\right) & \left(\begin{matrix}\overline{1\ldots n}\\ 2\ldots m\end{matrix}\right)\end{vmatrix},$$

by Theorem X,

$=\Delta_3$, by Theorem V.

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* See author's paper already referred to.

COROLLARY: $\binom{\overline{1\ldots n}}{1\ldots m}^{(m-2)}=\begin{vmatrix} K'_3 & K'_2 & K'_1 \\ K'_4 & K'_3 & K'_2 \\ K'_5 & K'_4 & K'_3 \end{vmatrix}$, $K'_i=K_{n-m+i}$, $m \not> n$.

Proof: $$\binom{\overline{1\ldots n}}{1\ldots m}^{(m-2)}=\begin{vmatrix} \underset{\alpha=1\ldots m-2}{K'_3} & \underset{\alpha=1\ldots m-1}{K'_2} & \underset{\alpha=1\ldots m}{K'_1} \\ \underset{\alpha=2\ldots m-2}{K'_4} & \underset{\alpha=2\ldots m-1}{K'_3} & \underset{\alpha=2\ldots m}{K'_2} \\ \underset{\alpha=3\ldots m-2}{K'_5} & \underset{\alpha=3\ldots m-1}{K'_4} & \underset{\alpha=3\ldots m}{K'_3} \end{vmatrix},$$ by Theorems XI and I.

By means of Theorem IV the missing α's may be introduced as in the proof of the corollary of Theorem X, completing the proof.

THEOREM XII. *If* $m \not> n$,

$$\binom{\overline{1\ldots n}}{1\ldots m}^{(r)}=\begin{vmatrix} \binom{\overline{1\ldots n}}{1\ldots r} & \binom{\overline{1\ldots n}}{1\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{1\ldots m} \\ \binom{\overline{1\ldots n}}{2\ldots r} & \binom{\overline{1\ldots n}}{2\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{2\ldots m} \\ \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ \binom{\overline{1\ldots n}}{m-r+1\ldots r} & \binom{\overline{1\ldots n}}{m-r+1\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{m-r+1\ldots m} \end{vmatrix}$$
$$=\text{ say } \Delta_{m-r+1}.$$

Proof: It will be assumed that a relation shown in Theorem X to be valid for $\binom{\overline{1\ldots n}}{1\ldots m}^{(m-1)}$ is valid for $\binom{\overline{1\ldots n}}{1\ldots m-1}^{(r)}$, namely:

$$\binom{\overline{1\ldots n}}{1\ldots m-1}^{(r)}=\begin{vmatrix} \binom{\overline{1\ldots n}}{1\ldots r} & \binom{\overline{1\ldots n}}{1\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{1\ldots m-1} \\ \binom{\overline{1\ldots n}}{2\ldots r} & \binom{\overline{1\ldots n}}{2\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{2\ldots m-1} \\ \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ \binom{\overline{1\ldots n}}{m-r\ldots r} & \binom{\overline{1\ldots n}}{m-r\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{m-r\ldots m-1} \end{vmatrix}$$
$$=\text{ say } \Delta_{m-r}.$$

$$\text{Then } \binom{\overline{1\ldots n}}{1\ldots m}^{(r)}=\binom{\overline{1\ldots n}}{m-r+1\ldots m}\binom{\overline{1\ldots n}}{1\ldots m-1}^{(r)}-\overline{X}_{m-r+1,\,m-1}$$
$$=(1-\phi)\binom{\overline{1\ldots n}}{m-r+1\ldots m}\binom{\overline{1\ldots n}}{1\ldots m-1}^{(r)},$$
by Theorem IX, corollary,
$$=(1-\phi)\binom{\overline{1\ldots n}}{m-r+1\ldots m}\Delta_{m-r}=\Delta_{m-r+1}, \text{ by Theorem V.}$$

COROLLARY: $$\left(\overline{\begin{matrix}1\ldots n\\1\ldots m\end{matrix}}\right)^{(r)}=\begin{vmatrix}K'_{m-r+1} & K'_{m-r} & \ldots K'_1\\ K'_{m-r+2} & K'_{m-r+1} & \ldots K'_2\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\\ K'_{2(m-r)+1} & K'_{2(m-r)} & \ldots K'_{m-r+1}\end{vmatrix}$$

where $K'_i=K_{n-m+i}$, $m \not> n$.

Proof: $$\left(\overline{\begin{matrix}1\ldots n\\1\ldots m\end{matrix}}\right)^{(r)}=\begin{vmatrix}\underset{\alpha=1\ldots r}{K'_{m-r+1}} & \underset{\alpha=1\ldots r+1}{K'_{m-r}} & \underset{\alpha=1\ldots m}{K'_1}\\ \underset{\alpha=2\ldots r}{K'_{m-r+2}} & \underset{\alpha=2\ldots r+1}{K'_{m-r+1}} & \underset{\alpha=2\ldots m}{K'_2}\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\\ \underset{\alpha=m-r+1\ldots r}{K'_{2(m-r)+1}} & \underset{\alpha=m-r+1\ldots r+1}{K'_{2(m-r)}} & \underset{\alpha=m-r+1\ldots m}{K'_{m-r+1}}\end{vmatrix},$$

by Theorems XII and I.

By means of Theorem IV the missing α's may be introduced as in the proof of the corollary of Theorem X, completing the proof.

THEOREM XIII. *For all values of n,*

$$\left(\overline{\begin{matrix}1\ldots n\\1\ldots m\end{matrix}}\right)^{(r)}=\begin{vmatrix}K_{n-r+1} & K_{n-r} & \ldots K_{n-m+1}\\ K_{n-r+2} & K_{n-r+1} & \ldots K_{n-m+2}\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\\ K_{n-2r+m+1} & K_{n-2r+m} & \ldots K_{n-r+1}\end{vmatrix}. \qquad \text{(A)}$$

Proof: Theorem XII, corollary gives the result if $m \not> n$. If $m>n$, $\left(\overline{\begin{matrix}1\ldots n\\1\ldots m\end{matrix}}\right)^{(r)}$ can be calculated by the method of this paper with the interchange of rows and columns. The result will be the Δ' of Theorem VI where $t=m-r+1$ and $s=n-r+1$. But when $m>n$ the determinant of Theorem XII, corollary becomes the Δ of Theorem VI with the same values of t and s. The theorem follows.

COROLLARY: *For all values of n,*

$$\left(\overline{\begin{matrix}1\ldots n\\1\ldots m\end{matrix}}\right)^{(r)}=\begin{vmatrix}\left(\overline{\begin{matrix}1\ldots n\\1\ldots r\end{matrix}}\right) & \left(\overline{\begin{matrix}1\ldots n\\1\ldots r+1\end{matrix}}\right) & \ldots\left(\overline{\begin{matrix}1\ldots n\\1\ldots m\end{matrix}}\right)\\ \left(\overline{\begin{matrix}1\ldots n\\2\ldots r\end{matrix}}\right) & \left(\overline{\begin{matrix}1\ldots n\\2\ldots r+1\end{matrix}}\right) & \ldots\left(\overline{\begin{matrix}1\ldots n\\2\ldots m\end{matrix}}\right)\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\\ \left(\overline{\begin{matrix}1\ldots n\\m-r+1\ldots r\end{matrix}}\right) & \left(\overline{\begin{matrix}1\ldots n\\m-r+1\ldots r+1\end{matrix}}\right) & \ldots\left(\overline{\begin{matrix}1\ldots n\\m-r+1\ldots m\end{matrix}}\right)\end{vmatrix}.$$

This follows from Theorem XIII by retracing the steps of the proof of Theorem XII, corollary.

Section 5.

THEOREM XIV. $\overline{\begin{pmatrix}1\ldots n\\1\ldots m\end{pmatrix}}^{(r)}$, *where each element* u_{ij} *is a function of degree* b *is* $\dfrac{\prod\limits_{i=0}^{i=m-r} {}_{n+m-r}C_{r-1+i}}{\prod\limits_{i=0}^{i=m-r} {}_{n+m-r}C_i} \cdot b^{(m-r+1)(n-r+1)}$.

Proof: In this case the a's may be taken each equal to b, and the α's each equal to zero. K_i then reduces to $[\sum\limits_1^n a_1a_2\ldots a_i]_{a_j=b}={}_nC_rb^i$, and $\overline{\begin{pmatrix}1\ldots n\\1\ldots m\end{pmatrix}}^{(r)}$ to

$$b^{(m-r+1)(n-r+1)}\cdot\Delta \text{ where } \Delta=\begin{vmatrix} {}_nC_{n-r+1} & {}_nC_{n-r} & \ldots & {}_nC_{n-m+1}\\ {}_nC_{n-r+2} & {}_nC_{n-r+1} & \ldots & {}_nC_{n-m+2}\\ \ldots & \ldots & \ldots & \ldots\\ {}_nC_{n-2r+m+1} & {}_nC_{n-2r+m} & \ldots & {}_nC_{n-r+1}\end{vmatrix}.$$

Next we apply the formula ${}_nC_i={}_nC_{n-i}$ to each of the elements and multiply the resulting determinant by unity in the form of the determinant

$$\begin{vmatrix} {}_{m-r}C_0 & 0 & \ldots & 0\\ {}_{m-r}C_1 & {}_{m-r-1}C_0 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots\\ {}_{m-r}C_{m-r} & {}_{m-r-1}C_{m-r-1} & \ldots & 1\end{vmatrix}.$$

This gives for the element in the h-th row and the k-th column,

$$\begin{aligned} {}_{n+m-r+1-h}C_{r-1-h+k} &= \frac{(n+m-r+1-h)!}{(r-1-h+k)!(n+m-2r+2-k)!}\\ &= \frac{(n+m-r+1-h)!\,{}_{r-2+k}P_{h-1}}{(r-2+k)!(n+m-2r+2-k)!},\quad {}_iP_0=1.\end{aligned}$$

The factorial number in the numerator is constant for a given row as is the denominator for a given column. Hence $\Delta=A\Delta'$, where

$$A=\frac{\sum\limits_{h=1}^{h=m-r+1}(n+m-r+1-h)!}{\sum\limits_{k=1}^{k=m-r+1}(r-2+k)!(n+m-2r+2-k)!} \text{ and } \Delta'=\begin{vmatrix} h=1\ldots m-r+1\\ {}_{r-2+k}P_{h-1}\\ k=1\ldots m-r+1\end{vmatrix}=|a_{h,k}|.$$

Let $\underset{k=2\ldots m-r+1}{D'a_{h,k}}=a_{h,k}-a_{h,k-1}$, $D'a_{h,1}=a_{h,1}$. Then $|D'a_{h,k}|=\Delta'$.

$$\begin{aligned} D'a_{h,k} &= {}_{r-2+k}P_{h-1}-{}_{r-3+k}P_{h-1}=[(r-2+k)-(r-1+k-h)]\,{}_{r-3+k}P_{h-2}\\ &=(h-1)\,{}_{r-3+k}P_{h-2},\end{aligned}$$

. . . .

Finally

$$D^{(k-1)}a_{h,k} \equiv D^{(k-2)}a_{h,k} - D^{(k-2)}a_{h,k-1} = (h-1)(h-2)\dots(h-k+1)_{r-1}P_{h-k}$$
$$= \begin{cases} (h-1)!, \text{ if } k=h \\ 0, \text{ if } k>h. \end{cases}$$

Hence
$$\Delta' \equiv \begin{vmatrix} 0! & 0 & 0 & 0 & \dots & 0 \\ & 1! & 0 & 0 & \dots & 0 \\ & & 2! & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & . \\ \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & (m-r)! \end{vmatrix} = \prod_{i=0}^{i=m-r} i!.$$

Hence
$$\Delta = A'\Delta = \frac{\prod\limits_{i=0}^{i=m-r}(n+m-r-i)!\prod\limits_{i=0}^{i=m-r} i!}{\prod\limits_{i=0}^{i=m-r}(r+i-1)!\prod\limits_{i=0}^{i=m-r}(n+m-2r+1-i)!} = \frac{\prod\limits_{i=0}^{i=m-r} {}_{n+m-r}C_{r-1+i}}{\prod\limits_{i=0}^{i=m-r} {}_{n+m-r}C_i}.$$

The theorem follows.

COROLLARY: $\overline{\begin{pmatrix} 1 \dots n \\ 1 \dots m \end{pmatrix}}^{(r)}$, *where each element* u_{ij} *is a function of degree* b, *is*

$$\frac{\prod\limits_{i=0}^{i=m-r} {}_{n+i}C_{n-r+1}}{\prod\limits_{i=0}^{i=m-r} {}_{n-r+1+i}C_{n-r+1}} \cdot b^{(m-r+1)(n-r+1)}.^{*} \tag{B}$$

Proof: By changing the form of two of the factorial numbers in the value of $A\Delta'$, and introducing the factor $[(n-r+1)!]^{(m-r+1)}$ in the numerator and the denominator

$$A\Delta' = \frac{\prod\limits_{i=0}^{i=m-r}(n+i)!}{\prod\limits_{i=0}^{i=m-r}(r-1+i)!(n-r+1)!} \cdot \frac{\prod\limits_{i=0}^{i=m-r} i!(n-r+1)!}{\prod\limits_{i=0}^{i=m-r}(n-r+1+i)!} = \frac{\prod\limits_{i=0}^{i=m-r} {}_{n+i}C_{n-r+1}}{\prod\limits_{i=0}^{i=m-r} {}_{n-r+1+i}C_{n-r+1}}.$$

The corollary follows.

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* See paper by Segre, already referred to.

On the Lie-Riemann-Helmholtz-Hilbert Problem of the Foundations of Geometry.

BY ROBERT L. MOORE.

§ 1. *Introduction.*

Concerning Hilbert's paper, "Über die Grundlagen der Geometrie,"* Poincaré says, according to Halsted's translation,† "As regards the ideas of Lie, the progress made is considerable. Lie supposed his groups defined by analytic equations; Hilbert's hypotheses are far more general. Without doubt this is still not entirely satisfactory, since though the *form* of the group is supposed any whatever, its *matter*, that is to say the plane which undergoes the transformations, is still subjected to being a *number-manifold* in Lie's sense. Nevertheless, this is a step in advance, and besides Hilbert analyzes better than anyone before him the idea of *number-manifold* and gives outlines which may become the germ of an assumptional theory of analysis situs."

The present paper contains a set of assumptions Σ in terms of the notions *point*, *region*, and *motion*. Here the space which undergoes the transformations (motions) is *not subjected in advance* to the condition of being a number plane nor is it *presupposed* that the *regions* are in one-to-one correspondence with portions of such a plane. Of course Poincaré's statement that "the form of the group is supposed any whatever" applies to its *presupposed* form. Hilbert's axioms so condition the form of the group in question as to necessitate that it should be simply isomorphic with the group of rigid motions in a space of two dimensions. It is largely, or entirely, a question of analysis. It may be said that Hilbert *analyzes* the group of transformations (motions) but leaves largely unanalyzed the space that undergoes the transformations. In the present treatment the "form" of the transformations and their "matter" (the space that is transformed by them) are subjected to what might be termed a *simultaneous analysis*.

* *Matematische Annalen*, Vol. LVI (1902–03), pp. 381–422.

† "The Bolyai Prize," *Science*, May 19, 1911, p. 765.

§ 2. *Preliminary Explanations and Definitions.*

I consider a class $\bar{S}$ of undefined elements called *points*, an undefined class of sub-classes of $\bar{S}$ called *regions* and an undefined class of one-to-one transformations of $\bar{S}$ into itself called *motions*.* If P is a point of $\bar{S}$, and M is a motion, the point into which P is transformed by M will be denoted by the symbol $M(P)$. If K is a point-set and M is a motion, $M(K)$ will denote the set of all points $M(P)$ for all points P of K.

DEFINITIONS. A point P is said to be a limit point of a point-set K if and only if every region that contains P contains at least one point of K distinct from P. The boundary of a point-set K is the set of all points $[X]$ such that every region that contains X contains at least one point of K and at least one point that does not belong to K. If K is a point-set, K' denotes the set of points composed of K plus its boundary. If R is a region the point-set $\bar{S}—R'$ is called the *exterior* of R. A point in the exterior of R is said to be *without* R.

A set of points is said to be *connected* if however it be divided into two mutually exclusive subsets, one of them contains a limit point of the other one. A set of points is said to be *closed* if it contains all its limit points. A set of points is said to be *continuous* if it is both closed and connected.

A *domain* is a set of points K such that if P is a point of K then there exists a region that contains P and is contained in K.

A set of regions G is said to *cover* a point-set K if each point of K belongs to at least one region of the set G. If for every infinite set of regions G that covers the point-set K there exists a finite subset of G that also covers K, then K is said to *possess the Heine-Borel property.*

A set of points K is said to be *bounded* if there exists a region R such that K is a subset of R'.

If A and B are two distinct points, a *simple continuous arc* from A to B is a continuous bounded point-set that contains A and B, but is disconnected† by the omission of any one of its points other than A and B.

A *simple closed curve* is a continuous bounded point-set which is disconnected by the omission of any two of its points.

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* By a one-to-one transformation of $\bar{S}$ into itself is meant a transformation T such that (1) for each point P of $\bar{S}$ there exists one and only one point $\bar{P}$ of $\bar{S}$ such that T transforms P into $\bar{P}$, (2) for each point $\bar{P}$ of $\bar{S}$ there is one and only one point P of $\bar{S}$ such that T transforms P into $\bar{P}$. *Point* is wholly undefined. *Region* is undefined except in so far as it is understood that every region is *some* sort of collection of points. *Motion* is undefined except in so far as it is understood that every motion is *some* sort of one-to-one transformation of $\bar{S}$ into itself. In addition to this information, no further information (aside from that furnished by the axioms of the system Σ) is presupposed concerning the terms *point, region,* and *motion.*

† A connected point-set K is said to be disconnected by the omission of a proper subset N if $K—N$ is not connected.

§ 3. *The Axioms of* Σ.

Axiom 1. *There exists at least one region.*

Axiom 2. *If R and K are regions and R' is a subset of K' then R is a subset of K.*

Axiom 3. *If the region R_1 contains the point O in common with the region R_2, there exists a region R containing O such that R' is common to R_1 and R_2.*

Axiom 4. *If R_1 and R_2 are regions and R_2' is a subset of R_1 then $R_1 - R_2'$ is a non-vacuous connected point-set.*

Axiom 5. *If R_1 and R_2 are regions there exists a region R that contains both R_1' and R_2'.*

Axiom 6. *Every simple closed curve is the boundary of a region.*

Axiom 7. *If O is a point and L and N are closed bounded point-sets with no point in common, there exists a region K containing O such that if P is a point in K then every region that contains both a point of L and a point of N can be transformed, by a motion that carries some point of L into O, into a point-set that contains both O and P.*

Axiom 8. *If R is a region and M is a motion then $M(R)$ is a region.*

Axiom 9.* *If A, B, C, A', B', C' are points, distinct or otherwise, such that every three regions that contain A, B and C, respectively, can be transformed by some motion into regions containing A', B', and C', respectively, then there exists a motion that transforms A into A', B into B', and C into C'.*

Axiom 10.† *If M is a motion there exists a motion M^{-1} such that if $M(A)=B$ then $M^{-1}(B)=A$.*

Axiom 11.‡ *If M and N are motions there exists a motion MN such that, for every point P, $M(N(P))=MN(P)$.*

Axiom 12.§ *If R_1 and R_2 are regions bounded respectively by the simple closed curves J_1 and J_2, R_1' and R_2' have no point in common, A_1, B_1, and C_1 are three distinct points on J_1, and A_2, B_2, and C_2 are three distinct points on J_2, and there exist three simple continuous arcs A_1XA_2, B_1YB_2, and C_1ZC_2 such that no two of these arcs have a point in common and no one of them has any point other than an end-point in common either with R_1' or with R_2' and M is a motion such that R_1' and $M(R_2')$ have no point in common and*

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* Cf. Hilbert's Axiom III, *loc. cit.*, p. 169.

† Cf. Hilbert, *loc. cit.*, p. 167.

‡ Cf. Hilbert's Axiom I, *loc. cit.*, p. 167.

§ Cf. J. R. Kline, "A Definition of Sense on Closed Curves in Non-metrical Plane Analysis Situs," *Annals of Mathematics*, Vol. XIX (1918), pp. 185–200. Axiom 12 corresponds to Hilbert's assumption (*loc. cit.*, p. 167) that motion does not change sense on any simple closed curve.

there exist three arcs $A_1\overline{X}M(A_2)$, $B_1\overline{Y}M(B_2)$, and $C_1\overline{Z}M(C_2)$ from A_1 to $M(A_2)$, from B_1 to $M(B_2)$, and from C_1 to $M(C_2)$, respectively, then there exist three such arcs such that no two of them have a point in common and no one of them has any point other other than end-point in common either with R_1' or with $M(R_2')$.

§ 4. *Consequences of Axioms 1–4 and 7–11.*

THEOREM 1. *If the point P is a limit point of the point-set K, and M is a motion, then $M(P)$ is a limit point of $M(K)$.*

Proof. If $M(P)$ were not a limit point of $M(K)$ there would exist a region R containing $M(P)$, but no point of $M(K)$ other than $M(P)$. But in this case $M^{-1}(R)$ would be a region containing P but no point of K other than P, contrary to the hypothesis that P is a limit point of K.

THEOREM 2. *No point of a region is a boundary point of that region.*

THEOREM 3. *Every region contains infinitely many points.*

Theorem 3 can be easily proved with the use of Axioms 3 and 4.

THEOREM 4. *If A and B are distinct points, and C is any point whatever, there exists a region containing C which can not be transformed by a motion into a point-set K such that K' contains both A and B.*

Proof. If no region contains the point C then it is vacuously true that if R_1, R_2, and R_3 are three regions containing C there exists a motion $\overline{M}$ such that $\overline{M}(R_1)$ contains A, and $\overline{M}(R_2)$ and $\overline{M}(R_3)$ contain B. It follows by Axiom 9 that there exists a motion that carries C into both A and B. Thus the supposition that there is no region containing C leads to a contradiction. Suppose that every region containing C can be transformed by a motion into a point-set K such that K' contains both A and B. By Axiom 3, if R is a region containing C, there exists a region $\overline{R}$ containing C such that $\overline{R}'$ is a subset of R. By hypothesis there exists a motion M such that $[M(\overline{R})]'$ contains both A and B. By Theorem 1 $[M(\overline{R})]'=M(\overline{R}')$. Hence $M(R)$ contains both A and B. It follows by Axiom 9 that there exists a motion that transforms C into both A and B. Thus the supposition that Theorem 4 is false leads to a contradiction.

THEOREM 5. *If L and N are two closed, bounded point-sets with no point in common, and O is any point whatever, there exists a region R containing O such that R' can not be transformed by a motion into a point-set that contains both a point of L and a point of N.*

Proof. By Axiom 7 and Theorem 3 there exists a point P distinct from O such that every region that contains both a point of L and a point of N can be transformed by a motion into a point-set that contains both O and P. By Theorem 4 there exists about* O a region $\bar{R}$ which can not be moved into a point-set containing both O and P. If there should exist a motion M such that $M(\bar{R})$ contains both a point of L and a point of N, then there would exist a motion $\bar{M}$ such that $\bar{M}M(\bar{R})$ contains both O and P. But this would involve a contradiction. By Axiom 3 there exists about O a region R such that R' is a subset of $\bar{R}$. The region R clearly satisfies the condition of Theorem 5.

THEOREM 6. *If P is a limit point of $M+N$ it is a limit point either of M or of N.*

Theorem 6 can be proved with the use of Axiom 3.

THEOREM 7. *If the point P is a limit point of the point-set M, then every region that contains P contains infinitely many points of M.*

Proof. Suppose the region R contains the point P and has in common with M only a finite set of points $P_1, P_2, P_3, \ldots, P_n$ distinct from P. By Theorem 4 for each $i(1 \leqq i \leqq n)$ there exists about P a region R_i that can not be transformed by a motion into a point-set containing P and P_i. But there exists a motion that leaves all points fixed. This motion carries R_i into R_i. But R_i contains P. It follows that R_i does not contain P_i. The regions R_1 and R_2 contain in common a region K_2 containing P. The region K_2 contains neither P_1 nor P_2. Similarly there exists in K_2 a region K_3 that contains P, but no one of the points P_1, P_2, P_3. This process may be continued. It follows that there exists about P a region K_n that lies in R but contains no point of the set $P_1, P_2, P_3, \ldots, P_n$. Hence P is not a limit point of M. Thus the supposition that Theorem 7 is false leads to a contradiction.

THEOREM 8. *If the region R contains a point O in the region K and a point P without K, then it contains a point on the boundary of K.*

Proof. By Axiom 3 there exists about O a region R_1 which is a subset both of R and of K. By Theorem 3 there exists in R_1 a point $\bar{P}$ distinct from O. By Theorem 7 and Axiom 3 there exists about O a region R_2 such that R_2' is a subset of $R_1 - \bar{P}$. By Axiom 4 $R - R_2'$ is connected. But it contains the point $\bar{P}$ in K and the point P without K. Hence it contains a point of the boundary of K.

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* In this connection "about" is synonymous with "containing."

THEOREM 9. *If O is a point there exists a countably infinite sequence of regions $R_1, R_2, R_3, \ldots$, such that* (1) *P is the only point they have in common,* (2) *if n is a positive integer and M is a motion such that $M(R_{n+1})$ contains O, then $M(R'_{n+1})$ is a subset of R_n,* (3) *if R is a region containing O there exists an n such that R_n is a subset of R.*

Proof. By Axiom 1 and Theorems 3 and 4 there exists a region K_1 containing O. By Axiom 3 there exists a region K_2 containing O such that K'_2 is a subset of K_1 and a region K_3 containing O such that K'_3 is a subset of K_2. It follows by two applications of Axiom 4 that $K_1 - K'_3$ is a connected point-set containing at least two distinct points. Let $\bar{P}$ denote a definite point of $K_1 - K'_3$. Then $\bar{P}$ is a limit point of $K_1 - K'_3 - \bar{P}$. It follows that there exists a countable sequence of points $\bar{P}_1, \bar{P}_2, \bar{P}_3, \ldots$ all distinct from $\bar{P}$ such that $\bar{P}$ is a limit point of the point-set $\bar{P}_1 + \bar{P}_2 + \bar{P}_3 + \ldots$. By Theorem 7 and Axiom 7 there exists a motion $\bar{M}$ such that $\bar{M}(\bar{P}) = O$. For every n let P_n denote $\bar{M}(\bar{P}_n)$. Then O is a limit point of the point-set $P_1 + P_2 + P_3 + \ldots$. By Theorem 4 there exists about O a region R_1 such that if M is a motion that transforms R_1 into a point-set containing O then $M(R'_1)$ does not contain P_1. By Theorems 6, 7 and 5 there exists a region R_2 containing O such that if M is a motion that transforms R_2 into a point-set containing O, then $M(R'_2)$ contains no point of the closed, bounded point-set $P_2 + L_1$ where L_1 is the boundary of R_1. It follows by Theorem 8 that, for every such motion M, $M(R'_2)$ is a subset of R_1. This process may be continued. It follows that there exists a sequence of regions $R_1, R_2, R_3, \ldots$ containing O such that if n is a positive integer and M is a motion that transforms R_{n+1} into a point-set containing O, then $M(R'_{n+1})$ contains no point of the point-set $P_1 + P_2 + P_3 + \ldots + P_n + S - R_n$. The sequence $R_1, R_2, R_3, \ldots$ satisfies the requirements of Theorem 9. That it satisfies requirement (2) is obvious. Suppose it does not satisfy both (1) and (3). Then there exists a point P distinct from O and a region R containing O such that for every n R_n contains a point of the closed and bounded point-set $P + L$ where L is the boundary of R. It follows by Axiom 7 that there exists a positive integer m such that, for every n, R_n can be transformed by some motion into a point-set containing P_m and O. But R_{m+1} can not be moved into such a point-set. Thus the supposition that $R_1, R_2, R_3, \ldots$ does not satisfy requirements (1) and (3) has led to a contradiction.

THEOREM 10. *Every region is a connected set of points.*

Proof. Suppose R_1 is a region. There exists in R_1 a point P. By Theorem 9 there exists a sequence of regions $R_2, R_3, R_4, \ldots$, all lying in R_1

and having in common only the point P and such that (1) for each n, R'_{n+1} is a subset of R_n, (2) if R is a region containing P there exists an n such that R contains R_n. By Axiom 4, $R_1-R'_n$ is connected. But

$$R_1=(R_1-R'_2)+(R_1-R'_3)+(R_1-R'_4)+\ldots.+P,$$

$R_1-R'_n$ is a subset of $(R_1-R'_{n+1})$ and P is a limit point of

$$(R_1-R'_2)+(R_1-R'_3)+(R_1-R'_4)+\ldots..$$

It easily follows that R_1 is connected.

THEOREM 11. *Every boundary point of a region is a limit point of the exterior of that region.*

Proof. Suppose the boundary of the region R contains a point X which is not a limit point of $\bar{S}-R'$. Then there exists a region $\bar{R}$ that contains X and lies wholly in R'. It follows that $\bar{R}'$ is a subset of R'. Therefore, by Axiom 2, $\bar{R}$ is a subset of R. Thus X belongs to R and is therefore not a boundary point of R.

THEOREM 12. *If $R_1, R_2, R_3, \ldots.$ is a sequence of regions closing down* on the point O, $M_1, M_2, M_3, \ldots.$ are motions and L and N are two closed and bounded point-sets with no point in common, then there do not exist infinitely many positive integers n such that $M_n(R'_n)$ contains both a point of L and a point of N.*

Theorem 12 is a consequence of Theorem 5.

THEOREM 13. *If $R_1, R_2, R_3, \ldots.$ is a sequence of regions closing down on the point O, $M_1, M_2, M_3, \ldots.$ are motions, and $A_1, A_2, A_3, \ldots.$ and $B_1, B_2, B_3, \ldots.$ are two infinite sequences of points such that, for every n, A_n and B_n are both in $M_n(R_n)$, then if $A_1+A_2+A_3+\ldots.$ has a limit point every such point is also a limit point of $B_1+B_2+B_3+\ldots..$*

Proof. Suppose X is a limit point of $A_1+A_2+A_3+\ldots.$, and R is a region containing X. By Axiom 3 there exists a region $\bar{R}$ containing X such that $\bar{R}'$ is a subset of R. The region $\bar{R}$ contains infinitely many distinct points $A_{n_1}, A_{n_2}, A_{n_3}, \ldots.$ of the sequence $A_1, A_2, A_3, \ldots..$ It follows with the help of Theorems 8 and 12 that, for infinitely many positive integers i, R_{n_i}, and therefore B_{n_i}, is a subset of R. It follows that X is a limit point of $B_1+B_2+B_3+\ldots..$

THEOREM 14. *Every bounded infinite set of points has at least one limit point.*

*A sequence of regions $R_1, R_2, R_3, \ldots.$ is said to *close down on* the point O if it satisfies with respect to O all the requirements of Theorem 9.

Proof. Suppose that R is a region and that $X_1, X_2, X_3, \ldots.$ is an infinite set of distinct points lying in R' and having no limit point. Let $R_1, R_2, R_3, \ldots.$ denote a sequence of regions closing down on a point O. For each n there exists a motion M_n such that $M_n(O) = X_n$. For each n there exists in the region $M_n(R_n)$ a point $\overline{X}_n$ distinct from every point of the set $X_1, X_2, X_3, \ldots.$ and lying in R'. By Theorem 13 the point-set $\overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \ldots.$ has no limit point. Thus $X_1 + X_2 + X_3 + \ldots.$ and $\overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \ldots.$ are closed, bounded point-sets with no point in common. But, for every n, $M_n(R_n)$ contains a point of each of these sets. This is contrary to Theorem 12.

THEOREM 15. *There does not exist a compact* point-set K and an uncountably infinite set G of mutually exclusive regions such that every region of the set G contains a point of K.*

Proof. Suppose there does exist a compact point-set K and an uncountable set of regions G satisfying such conditions. Let $R_1, R_2, R_3, \ldots.$ denote a sequence of regions closing down on some point O. If X is a point of K lying in a region g of the set G then for each n there exists a motion M_n such that $M_n(O) = X$. By Theorems 8 and 12 there exists a positive integer m such that $M_m(R_m)$ is a subset of g. It follows by Zermelo's Postulate that there exists a set $\overline{G}$ of regions such that (1) each region of the set G contains one and only one region of the set $\overline{G}$, (2) for each region $\overline{g}$ of the set $\overline{G}$ there exists a motion that transforms some region of the set $R_1, R_2, R_3, \ldots.$ into $\overline{g}$ and transforms the point O into a point of K that lies in $\overline{g}$. In view of the fact that G is an uncountable set it follows that there exists a positive integer n and a set of motions $M_1, M_2, M_3, \ldots.$ such that, for every j, $M_j(O)$ is a point of K and such that no two of the regions $M_1(R_n), M_2(R_n), M_3(R_n), \ldots.$ have a point in common. By hypothesis the set of points $M_1(O) + M_2(O) + M_3(O) \ldots.$ has at least one limit point Z. There exists a motion $\overline{M}$ such that $\overline{M}(R_{n+1})$ contains Z. There exists k such that $M_k(O)$ is in $\overline{M}(R_{n+1})$. Hence O is in $M_k^{-1}\overline{M}(R_{n+1})$. It follows that $M_k^{-1}\overline{M}(R'_{n+1})$ is a subset of R_n. Hence $\overline{M}(R'_{n+1})$ is a subset of $M_k(R_n)$. Therefore $M_k(R_n)$ contains Z. Hence there exists an index i distinct from k such that $M_k(R_n)$ contains $M_i(O)$. It follows that $M_k(R_n)$ and $M_i(R_n)$ have a point in common. Thus the supposition that Theorem 15 is false leads to a contradiction.

THEOREM 16. *Every compact set of points is a subset of a compact domain.*

* A set of points K is said to be *compact* if every infinite subset of K has at least one limit point. Cf. M. Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del circolo matematico di Palermo*, Vol. XXII (1906), p. 6.

Proof. Let $R_1, R_2, R_3, \ldots$ be a sequence of regions closing down on a point O. There exists an index m greater than 1 such that if M is a motion such that $M(R_m)$ contains a point of R_1 then $M(R_m)$ is a subset of R_1. Suppose there exists a compact set of points K which is not a subset of a compact domain. Then there must exist an infinite set of distinct points $P_1, P_2, P_3, \ldots$ lying in K, an infinite set of distinct points $\overline{P}_1, \overline{P}_2, \overline{P}_3, \ldots$ such that $\overline{P}_1+\overline{P}_2+\overline{P}_3+\ldots$ has no limit point and a set of motions $M_1, M_2, M_3, \ldots$ such that for every n the region $M_n(R_m)$ contains both P_n and $\overline{P}_n$. The point-set $P_1+P_2+P_3$ has at least one limit point $\overline{O}$. There exists a motion $\overline{M}$ such that $\overline{M}(O)=\overline{O}$. There exists an infinite set of distinct positive integers $n_1, n_2, n_3, \ldots$ such that the points $P_{n_1}, P_{n_2}, P_{n_3}, \ldots$ are all in $\overline{M}(R_m)$. Each of the regions $M_{n_1}(R_m), M_{n_2}(R_m), M_{n_3}(R_m), \ldots$ is a subset of $\overline{M}(R_1)$. Hence the infinite point-set $\overline{P}_{n_1}+\overline{P}_{n_2}+\overline{P}_{n_3}+\ldots$ is a subset of $\overline{M}(R_1)$. It follows by Theorem 14 that $\overline{P}_1+\overline{P}_2+\overline{P}_3+\ldots$ has a limit point. Thus the supposition that Theorem 16 is false has led to a contradiction.

THEOREM 17. *If the compact point-set T is covered by an uncountably infinite set of regions G, then T is covered by some countable subset of G.*

Proof. By Theorem 16 the compact point-set T is a subset of some compact domain K. Let $R_1, R_2, R_3, \ldots$ denote a sequence of regions closing down on a point O. For each point P of T there exists an integer n_P greater than 1 and a motion M_P such that the region $M_P(R_{n_P-1})$ contains P and lies both in K and in some region of the set G. Let $\overline{G}$ be the set of all regions $M_P(R_{n_P})$ for all points P of T. There exists a finite or countably infinite set of distinct positive integers $m_1, m_2, m_3, \ldots$ and a sequence $\overline{G}_{m_1}, \overline{G}_{m_2}, \overline{G}_{m_3}, \ldots$ of subsets of $\overline{G}$ such that (1) $\overline{G}=\overline{G}_{m_1}+\overline{G}_{m_2}+\overline{G}_{m_3}+\ldots$, (2) for every i $\overline{G}_{m_i}$ is the set of all regions $M_P(R_{n_P})$ for which $n_P=m_i$. For every i let T_i denote the set of all those points of T which are covered by the set of regions $\overline{G}_{m_i}$. The regions of the set $\overline{G}_{m_1}$ can be arranged in a well-ordered sequence $g_1, g_2, g_3, \ldots g_\omega, \ldots$. Call this well-ordered sequence β. Let g_2 be the first region in the sequence β that contains a region that has no point in common with g_1. Let g_3 be the first one following g_2 that contains a region that has no point in common with g_1 or with g_2. Continue this process, thus obtaining a well-ordered sequence γ such that (1) every element of γ is an element of β, (2) if γ_1 is any subsequence of γ such that there exists an element of γ that follows all the elements of γ_1 then the first element of γ that follows all the elements of γ_1 is the first element of β that contains a region that has no point in common with any of the elements of γ_1. There is not an uncountable

infinity of elements in the sequence γ. For if there were, there would exist in K an uncountable infinity of distinct regions, no two of which have a point in common, which is contrary to Theorem 15. It follows that there exists a countable subset G_i of the set of regions $\overline{G}_{m_i}$ such that every point of T_i either is in a region of the set G_i or is a limit point of the point-set L_i obtained by adding together all the regions $M_{P_1}(R_{m_i})$, $M_{P_2}(R_{m_i})$, $M_{P_3}(R_{m_i})$, ... of the set G_i. Let $\overline{\overline{G}}_i$ denote the set of regions $M_{P_1}(R_{m_i-1})$, $M_{P_2}(R_{m_i-1})$, $M_{P_3}(R_{m_i-1})$, ... The set $\overline{\overline{G}}_i$ covers T_i. For suppose there exists a point X of T_i which lies in no region of the set $\overline{\overline{G}}_i$. Then X is a limit point of L_i. There exists a positive integer k_i such that the region R_{k_i} can not be transformed by a motion into a point-set containing both a point in R_{m_i} and a point in $S-R_{m_i-1}$. But there exists a motion $\overline{M}$ such that $\overline{M}(R_{k_i})$ contains X. The region $\overline{M}(R_{k_i})$ contains a point of L_i and therefore, for some j, a point in common with $M_{P_j}(R_{m_i})$. If it also contained a point in common with $\overline{S}-M_{P_j}(R_{m_i-1})$ then $M_{P_j}^{-1}\overline{M}(R_{k_i})$ would contain a point of R_{m_i} and a point of $\overline{S}-R_{m_i-1}$. But this is impossible. It follows that, for every i, T_i is covered by $\overline{\overline{G}}_i$. But every region of $\overline{\overline{G}}_i$ is a subset of some region of G, and T is the sum of the finite or countably infinite set of point-sets T_1, T_2, T_3, ... It follows that T is covered by a countable subset of G.

THEOREM 18. *If a closed and compact point-set is covered by an infinite set of regions then it is also covered by some finite subset of that set of regions.*

Proof. By Theorem 17 if a compact point-set is covered by an infinite set of regions G it is covered by a countable subset of G. But if a closed and compact point-set is covered by a *countable* set of regions then* it is covered by a finite subset of that countable set.

THEOREM 19. *If H' is a closed and bounded set of points there exists an infinite sequence of regions $K_{H'1}$, $K_{H'2}$, $K_{H'3}$, ... such that (1) if m is a positive integer and P is a point of H', there exists an integer n, greater than m, such that $K_{H'n}$ contains P, (2) if P and $\overline{P}$ are distinct points of H' lying in a region R, there exists an integer δ such that if $n>\delta$ and $K_{H'n}$ contains P then $K_{H'n}$ is a subset of $R-\overline{P}$.*

Proof. Let R_1, R_2, R_3, ... be a set of regions closing down on a point O. If X is a point of H' and n is a positive integer, the region R_n can be transformed by a motion into a region K_{Xn} containing X. For any fixed n consider

* Cf. F. Hausdorff, "Grundzüge der Mengenlehre," Veit & Co., Leipzig 1914, p. 231.